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INVESTIGATION OF THE VALIDITY OF THE
NON-ROTATING PLANET ASSUMPTION FOR
THREE-DIMENSIONAL EARTH ATMOSPHERIC ENTR

THESIS

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THESIS

Presented to the Faculty of the School of Engineering
of the Air Force Institute of Technology
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Master of Science in Astronautical Engineering

Harry A. Karasopoulos, B.S.

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Preface

My purpose of study was to examine the non-rotating planet assumption for three-dimensional Earth atmospheric entry. Having worked in the area of hypersonic vehicle performance analysis for the Air Force Flight Dynamics Laboratory, I have become familiar with a variety of atmospheric entry analysis methods. It is obvious that the Earth's rotation needs to be accounted for in general entry studies. However, I was curious to see if trajectory states existed where the non-rotating planet equations of motion were equivalent to the rotating planet equations of motion. This thesis is the result of this curiosity.

I am deeply indebted to my faculty advisor, Capt. Rodney Bain, for his enthusiastic assistance, his patience, and for the many hours of instruction on perturbation theory he generously gave me. I thank Dr. L. E. Miller, formerly of the Air Force Flight Dynamics Laboratory, for his assistance, encouragement, and unique outlook. Special thanks is also owed to Dr. N. X. Vinh for his outstanding works in analytical flight mechanics that I heavily utilized in this paper and in the course of my Air Force civilian career. Finally, gratitude is expressed to my understanding fiancee, Susan, and my family and friends, who gave me motivational support throughout my AFIT experience.

Harry Karasopoulos

Table of Contents

List of Figures

Figure | Page

List of Tables

Notation

Roman Letter Symbols

a - Acceleration (ft/s^2)

B - Ballistic coefficient

C_D - Drag coefficient

C_L - Lift coefficient

D - Drag (lb)

g - Acceleration of gravity (ft/s^2)

g_* - Gravitational acceleration at the planet's surface (ft/s^2)

h - Non-dimensional altitude

I - Orbital inclination (deg, rad)

L - Lift (lb)

L/D - Lift-to-drag ratio

M - Mach number

m - Vehicle mass (lb_m)

q - Cosine of the flight path angle

r - Radius from center of planet (ft)

r_* - Planetary radius (ft)

S - Aerodynamic reference area (ft^2)

t - Time (s)

u - Speed ratio, a modified Chapman variable

V - Velocity (ft/s^2)

y - Altitude (ft)

Greek Letter Symbols

α - Longitude of the ascending node (deg, rad)

β - Inverse atmospheric scale height (1/ft)

γ - Flight path angle (deg, rad)

ϵ - Small parameter

ϵ_1 - Small parameter

ϵ_2 - Small parameter

θ - Longitude (deg, rad)

μ - Planet graviational parameter (ft^3/s^2)

ξ - Magnified non-dimensional altitude

ρ - Density (lb_m/ft^3)

σ - Bank angle (deg, rad)

ϕ - Latitude (deg, rad)

ψ - Heading angle (deg, rad)

ω - Planet rotation rate (deg/s, rad/s)

Ω - Argument of latitude at epoch (deg, rad)

Ω - Angular velocity (deg/s, rad/s)

Subscripts

$_*$ - At the surface of the planet

Superscripts

^{i} - Inner

^{c} - Composite

^{o} - Outer

$\dot{\ }$ – Time derivative

$\tilde{\ }$ – Unit direction vector

$\hat{\ }$ – Unit direction vector

Abstract

The assumption of a non-rotating planet, common in most analytical entry trajectory analyses, has been shown to produce significant errors in some solutions for the lifting atmospheric entry of Earth. This thesis presents an investigation of the validity of the non-rotating planet assumption for general three-dimensional Earth atmospheric entry.

In this effort, the three-dimensional equations of motion for lifting atmospheric entry are expanded to include a rotating planet model. A strictly exponential atmosphere, rotating at the same rate as the planet, is assumed with density as a function of radial distance from the planet's surface. Solutions are developed for the non-rotating Earth equations of motion and for one of the rotating Earth equations of motion using the method of matched asymptotic expansions.

It is shown that the non-rotating Earth assumption produces incorrect entry trajectory results for entry orbital inclination angles between 0.5 and 75.0 degrees and vehicle speeds ranging from circular orbital velocities to low supersonic speeds. However, a variety of realistic trajectory states exist where some of the non-rotating Earth equations of motion are valid. Three of the non-rotating equations of motion are found to be valid for the same entry trajectory states. Other, independent trajectory states exist where a fourth non-rotating Earth equation of motion is valid. A fifth equation of motion is never valid for the ranges of orbital inclination angle and speeds investigated. Trends in the results of the trajectory states of validity are discussed and methods to estimate some of these states are presented.

INVESTIGATION OF THE VALIDITY OF THE NON-ROTATING PLANET ASSUMPTION FOR THREE-DIMENSIONAL EARTH ATMOSPHERIC ENTRY

I. Introduction

Analytical studies often have the advantage of displaying valuable solution trends, giving more insight to the problem and its solutions than corresponding numerical work. Simple and accurate analytical methods to find solutions to the equations of motion for high speed flight vehicles supplement more complex and unwieldy numerical methods. In past analytical work on lifting atmospheric entry, the limiting assumptions of planar entry and a non-rotating planet were common. The equations of motion for planar atmospheric entry of a non-rotating planet are highly nonlinear; adding rotating planet and non-planar effects to the equations of motion make them even more difficult to solve analytically. Hence, the current literature has no investigations which analytically solves the equations of motion for three-dimensional, lifting atmospheric entry of a rotating planet.

The Non-Rotating Planet Assumption

Although common in most analytical entry trajectory analyses, the assumption of a non-rotating planet model can produce significant errors in trajectory results. Since most planets have a significant rotation rate, the rotating planet assumption will give more accurate entry vehicle performance results, especially for lifting vehicle range and time of flight calculations. Inherent in the concept of a rotating planet for atmospheric entry is the assumption that the planet's atmosphere rotates with the planet at a constant rate. This assumption is more accurate than the non-rotating planet/atmosphere assumption but is still not ideal. The atmosphere of a rotating planet can be viewed as a boundary layer with rotation rates which change with altitude. Near the planet's surface the atmosphere rotates at about the same rate as the planet. As altitude increases, the atmosphere rotates with a decreasing rate, and at high altitudes this rate is near zero. Hence, the true effect of a rotating atmosphere is therefore bounded on one end by the non-rotating planet solutions and on the other end by the rotating planet solutions. In this study is is assumed that the rotating planet solutions are ideally valid.

The maximum effect of the rotating atmosphere (Chapman, 1959:5) on a flight vehicle is easily calculated for a spherical planet. This maximum occurs at the equator for minimum altitude and is given by the ratio of the planet's angular velocity to the circular orbital velocity at the planet's surface. The planet's rotational velocity on the equator is given by

$$V_r = \omega r_*$$

where ω is the planet's rotation rate and r_* is the radius of the planet. The circular orbital velocity at the surface of the planet is

$$V_c = (g_* r_*)^{1/2}$$

where g_* is the gravitational acceleration at the surface of the planet. The ratio is given by

$$\left.\frac{V_r}{V_c}\right|_{r_*} = \left[\frac{\omega^2 r_*}{g_*}\right]^{1/2}$$

This ratio gives an indication of the possible error introduced to velocity calculations for a non-rotating planet model. For Earth, this ratio is approximately 0.06 . For Mars, Jupiter, Saturn, and Venus, this ratio is approximately 0.07, 0.30, 0.40, and 0.0, respectively (Vinh and others, 1980:3). Hence, for near-equatorial atmospheric entry, the maximum velocity error is negligible for Venus, significant for Earth and Mars, and very large for Jupiter and Saturn. The velocity error associated with the non-rotating planet assumption causes an even larger error in the calculation of aerodynamic forces. Since lift and drag are proportional to the square of velocity (as seen in Section II), the error introduced to the aerodynamic force calculations can be a maximum of about 0.14, 0.12, 0.60, and 0.80, for Mars, Earth, Jupiter, and Saturn, respectively.

A recent study of AOTV (Aero-assisted Orbital Transfer Vehicles) trajectories about Earth investigated possible trajectory simulation error due to the non-rotating planet assumption. In this study it was found that the non-rotating planet model caused velocity errors which gave dynamic pressure differences ranging up to 10 - 14% . These differences may cause underprediction of the final altitude and overprediction of the attainable orbital inclination change in a non-rotating Earth analysis. It was concluded that rotating Earth effects must be included for realistic AOTV trajectory simulation (Ikawa, 1986:1,9).

Another recent study (Miller, 1986:14) noted significant errors in values of range and time of flight for equilibrium glide entry trajectories when the Earth's rotation was neglected. Differences in trajectory results between the rotating and non-rotating cases were found to be significant, especially for trajectories beginning at speeds greater than 15,000 ft/s.

Vinh (Vinh and others, 1980:3) argues that inclusion of a rotating atmosphere into an analytical entry study would cause unwarranted overcomplication to the problem, possibly overshadowing the advantages of an analytical versus numerical analysis. However, for many atmospheric entry trajectories, such as multiple pass aerobraking, synergistic orbital plane change, and general high L/D vehicle trajectories with shallow entry flight paths, trends in the solutions caused by the rotating planet and its atmosphere may be important. For these types of trajectories, the error in the calculation of the aerodynamic forces is more prominent due to the relatively large flight times within the sensible atmosphere.

Problem

Because of their complexity, the equations of motion for three-dimensional, lifting atmospheric entry of a rotating planet have not been analytically solved. However, the assumption of a non-rotating planet, common in most analytical entry analyses, has been shown to produce significant errors in some solutions for the lifting atmospheric entry of Earth. An investigation of the general validity of the non-rotating planet assumption for three-dimensional Earth atmospheric entry is needed. In addition, the existence of trajectory states where the rotating planet terms in the equations of motion are negligible should be investigated. This would indicate the existence of trajectory states where existing solutions to the non-rotating equations of motion are valid for Earth lifting atmospheric entry.

Scope

In this effort, the three-dimensional exact equations of motion for lifting atmospheric entry are expanded to include a rotating planet model. The rotating planet terms in the equations of motion for Earth atmospheric entry are examined. Solutions are developed for the non-rotating equations of motion and for one of the rotating Earth equations of motion. This was accomplished by treating atmospheric entry as a boundary layer problem, and applying method of directly matched asymptotic expansions. A variety of realistic Earth entry trajectory states are shown to exist where some of the non-rotating equations of motion are valid for a rotating Earth. This validity is coordinate dependent since sigularities exist in the equations of motion. Entry trajectory state examination is limited to orbital inclination angles between 0.5 and 75.0 degrees, where most Earth atmospheric entry occurs, and vehicle speeds ranging from circular orbital velocity to low supersonic speeds where terminal maneuvers, such as landing approaches, are usually initiated.

Assumptions

The planet is modelled in this analysis by a sphere having a central gravitational force field obeying the inverse square law. A strictly exponential atmosphere, rotating at the same rate as the planet, is assumed with density as a function of radial distance from the planet's surface. In this effort, the only forces acting on the lifting vehicle are assumed to be gravity, lift, and drag; magnetic, solar wind, and other forces are considered negligible. The lifting entry vehicle is modelled as a point mass in a three degrees-of-freedom analysis. Constant lift-to-drag ratio is assumed along the trajectory and a ballistic coefficient is specified for each flight vehicle/atmosphere under study. Angle of attack was not explicitly modelled. More detailed discussion of the approximations and assumptions will be presented in Section III.

Approach

In Section II, the equations of motion for three-dimensional lifting entry for a spherical, rotating planet are derived. In Section III these equations of motion are transformed into a form more convenient to examine and solve. The equations are also made non-dimensional and a coordinate transformation is undertaken. In Section IV the rotating planet terms in each of the five equations of motion are examined. These terms then are set equal to zero and checked for the existence of real solutions. It is shown that three of the equations of motion have identical solutions for these rotating terms. It is also shown that real solutions do not exist for the rotating terms in one of the equations of motion. The solution to the equation of motion containing these rotating terms is developed in Section V along with the solutions to the non-rotating Earth equations of motion. These solutions are derived from the rotating Earth equations of motion using the method of matched asymptotic expansions. In Section VI, the solutions to the rotating term equations in the other four equations of motion are examined in more detail. Trajectory states are presented where some of the non-rotating Earth equations of motion are independently valid for a rotating Earth. Methods are given to estimate solutions where four of the non-rotating equations of motion are valid for rotating Earth entry. Conclusions and recommendations for further study are presented in Section VII.

II. Derivation of the Equations of Motion

In this section, the equations of motion are derived for three-dimensional lifting entry of a rotating planet. A spherical, rotating planet model is employed and it is assumed that the atmosphere rotates at the same rate as the planet with rotation rate, ω. The lifting entry vehicle is modelled as a point mass in a three degrees-of-freedom analysis. Gravity, lift, and drag are assumed to be the only forces acting on the vehicle; magnetic, solar wind, and other forces are assumed to be negligable. Further discussion of assumptions and approximations is presented in Section III.

Coordinate Systems

Figure 1 defines the planet centered coordinate systems used in this analysis. The planet's inertial reference frame has coordinates X, Y, and Z, with unit vectors $\tilde{I}$, $\tilde{J}$, and $\tilde{K}$, respectively. The X and Y axes lie in the planet's equatorial plane and the planet rotates about the Z axis. The rotating planet-fixed coordinate system has axes X_o, Y_o, and Z_o, with unit vectors $\tilde{I}_o$, $\tilde{J}_o$, and $\tilde{K}_o$, respectively. Another view of the planet centered coordinate systems is illustrated in Figure 2. $\hat{a}$ is a unit vector in the XY plane, pointing radially away from the planet. θ is defined

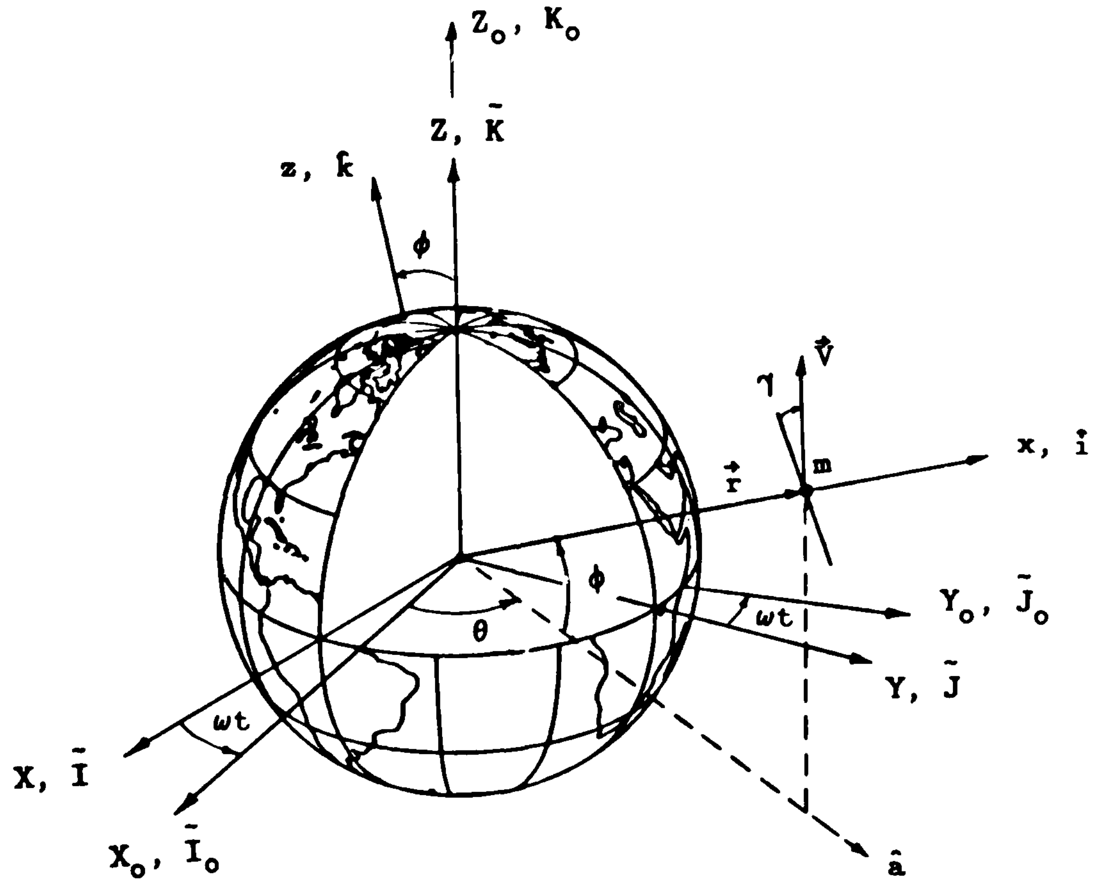

Figure 1. Planet Centered Coordinate Systems

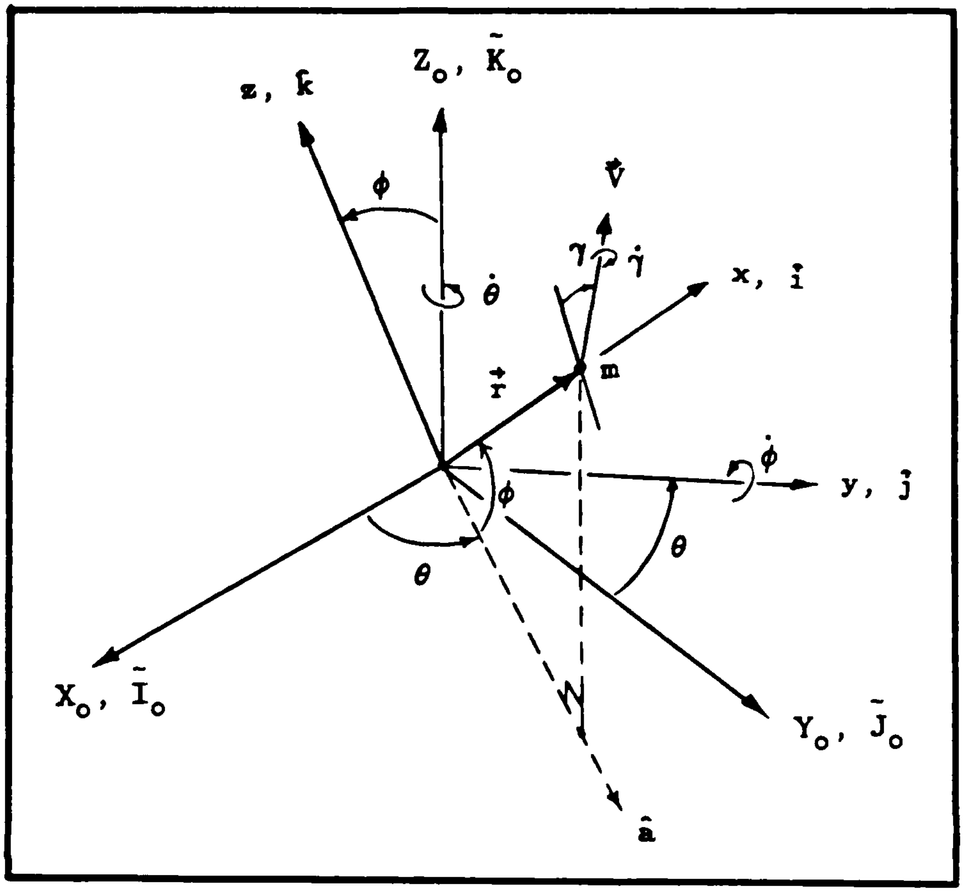

Figure 2. Coordinate Systems Revisited

as the longitude and is measured from the X_o axis which rotates about the $\tilde{K}_o$ direction with rate ω. ϕ is latitude and is measured positive from the equator to the pole in the $\tilde{K}_o$ direction. γ is the flight path angle and is measured "positive up" from the local horizontal to the velocity vector. The heading angle, ψ, is measured from a constant

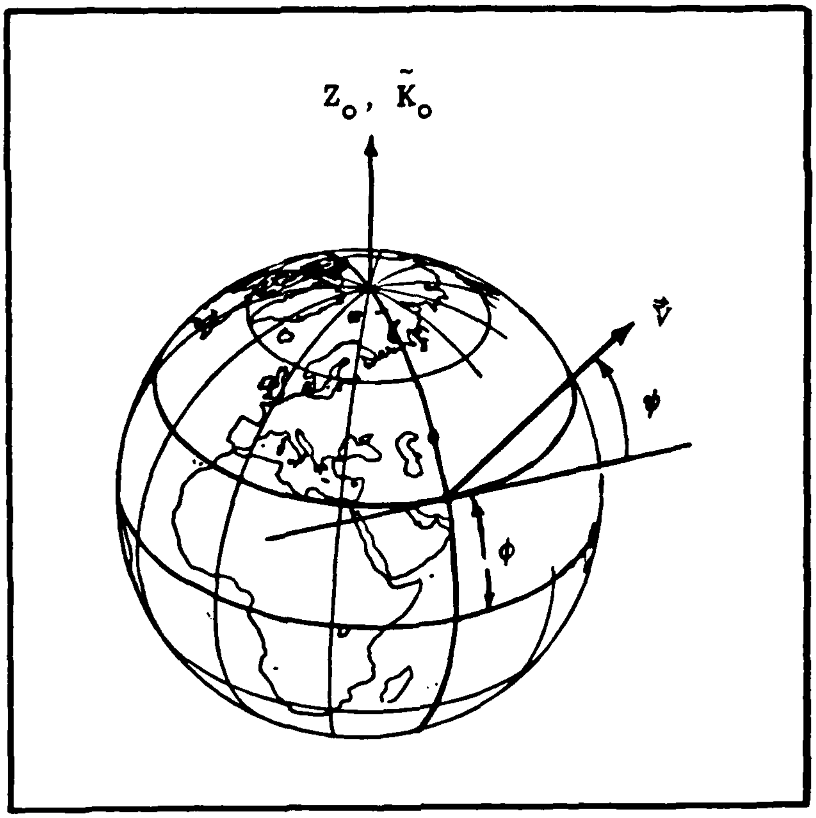

Figure 3. Heading Angle Definition

latitude line to the projection of the velocity vector onto the sphere, positive towards the $\tilde{K}_o$ direction (Figure 3). The vehicle has mass, m, and is at a radius, r from the center of the planet. Figure 4 presents the vehicle centered coordinate system.

Kinematic Equations of Motion

In order to develop the equations of motion, we first look at the velocity and acceleration equations for rotating

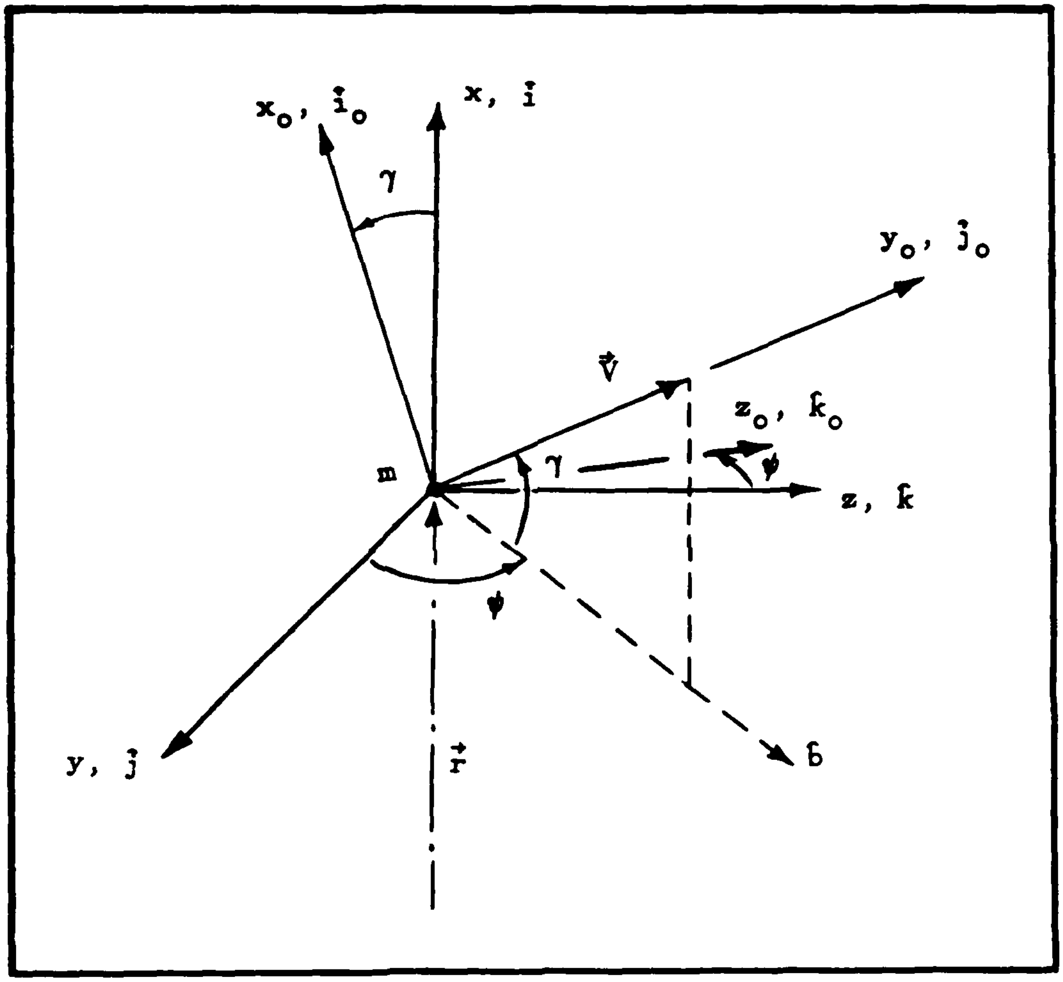

Figure 4. Vehicle Centered Coordinate Systems

systems. The velocity of the vehicle with respect to the inertial frame, X, Y, Z, is defined as the sum of the velocity of the vehicle with respect to the rotating frame and the cross product of the angular velocity of the rotating frame and the radius vector.

$$\vec{V}\Big|_{\tilde{I}\tilde{J}\tilde{K}} = \frac{d\vec{r}}{dt}\Big|_{\tilde{I}\tilde{J}\tilde{K}} = \frac{d\vec{r}}{dt}\Big|_{\hat{i}\hat{j}\hat{k}} + \vec{\Omega} \times \vec{r}\Big|_{\hat{i}\hat{j}\hat{k}}$$

$$= \vec{V}\Big|_{\hat{i}\hat{j}\hat{k}} + \vec{\Omega} \times \vec{r}\Big|_{\hat{i}\hat{j}\hat{k}} \qquad (2.10)$$

where $\vec{r}$ is the radius vector, extending from the planet's center to the flight vehicle, $\vec{V}$ is the velocity vector, and $\vec{\Omega}$ is the angular velocity of the rotating frame.

The inertial acceleration, a, is defined as the derivative of the inertial velocity.

$$\vec{a}\Big|_{\tilde{I}\tilde{J}\tilde{K}} = \frac{d^2\vec{r}}{dt^2}\Big|_{\tilde{I}\tilde{J}\tilde{K}} = \frac{d^2\vec{r}}{dt^2}\Big|_{\hat{i}\hat{j}\hat{k}} + \vec{\Omega} \times \frac{d\vec{r}}{dt}\Big|_{\hat{i}\hat{j}\hat{k}} + \frac{d\vec{\Omega}}{dt} \times \vec{r}\Big|_{\hat{i}\hat{j}\hat{k}}$$

$$+ (\vec{\Omega} \times \vec{\Omega}) \times \vec{r}\Big|_{\hat{i}\hat{j}\hat{k}} + \vec{\Omega} \times \frac{d\vec{r}}{dt}\Big|_{\hat{i}\hat{j}\hat{k}} + \vec{\Omega} \times \left[\vec{\Omega} \times \vec{r}\Big|_{\hat{i}\hat{j}\hat{k}}\right]$$

This can be simplified to the following

$$\vec{a}\Big|_{\tilde{I}\tilde{J}\tilde{K}} = \vec{a}\Big|_{\hat{i}\hat{j}\hat{k}} + 2\vec{\Omega} \times \frac{d\vec{r}}{dt}\Big|_{\hat{i}\hat{j}\hat{k}} + \frac{d\vec{\Omega}}{dt} \times \vec{r}\Big|_{\hat{i}\hat{j}\hat{k}}$$

$$+ \vec{\Omega} \times \left[\vec{\Omega} \times \vec{r}\Big|_{\hat{i}\hat{j}\hat{k}}\right] \qquad (2.12)$$

where

$\vec{a}\Big|_{\hat{I}\hat{J}\hat{K}}$ is the acceleration of the vehicle with respect to the inertial planet centered coordinate system.

$\vec{a}\Big|_{\hat{i}\hat{j}\hat{k}}$ is the acceleration of the vehicle with respect to the radius vector.

$\vec{\Omega}$ is the angular velocity of the rotating frame. For a point mass vehicle in flight over a rotating planet, $\vec{\Omega}$ is equal to a constant, the planet's rotation rate, $\vec{\omega}$.

$2\vec{\Omega} \times \dot{\vec{r}}\Big|_{\hat{i}\hat{j}\hat{k}}$ is the Coriolis acceleration.

$\vec{\Omega} \times \left[\vec{\Omega} \times \vec{r}\Big|_{\hat{i}\hat{j}\hat{k}}\right]$ is the Centripetal or Transport acceleration.

To apply Eq (2.12) in the derivation of the equations of motion, transformations between the various coordinate systems are required. The coordinate transformation from the inertial system, XYZ, to the vehicle centered system, xyz, is not difficult but is prone to algebraic error because of the many intermediate steps and variables involved. For ease in derivation, this transformation of coordinate systems is divided into a number of simple angle rotations.

<u>1st Rotation, Inertial to Rotating</u>. The first rotation is the planet rotation about the Z axis. The rotation angle at time t is ωt. From Figure 1 it can be seen that the following expressions apply:

$$\tilde{I}_o = \tilde{I}\cos\omega t + \tilde{J}\sin\omega t$$

$$\tilde{J}_o = -\tilde{I}\sin\omega t + \tilde{J}\cos\omega t$$

$$\tilde{K}_o = \tilde{K}$$

In matrix form

$$\begin{bmatrix} \tilde{I}_o \\ \tilde{J}_o \\ \tilde{K}_o \end{bmatrix} = \begin{bmatrix} \cos\omega t & \sin\omega t & 0 \\ -\sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{I} \\ \tilde{J} \\ \tilde{K} \end{bmatrix} \tag{2.1}$$

and

$$\begin{bmatrix} \tilde{I} \\ \tilde{J} \\ \tilde{K} \end{bmatrix} = \begin{bmatrix} \cos\omega t & -\sin\omega t & 0 \\ \sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{I}_o \\ \tilde{J}_o \\ \tilde{K}_o \end{bmatrix} \tag{2.2}$$

<u>2nd Rotation, Longitude</u>. The second rotation (Figure 5) is a longitude angle rotation about the Z axis:

$$\hat{a} = \tilde{I}_o\cos\theta + \tilde{J}_o\sin\theta \qquad \text{and} \qquad \hat{j} = -\tilde{I}_o\sin\theta + \tilde{J}_o\cos\theta$$

<u>3rd Rotation, Latitude</u>. The third rotation is a latitude angle rotation about the y axis. From Figure 6

$$\hat{i} = \hat{a}\cos\phi + \tilde{K}_o\sin\phi \qquad \text{and} \qquad \hat{k} = -\hat{a}\sin\phi + \tilde{K}_o\cos\phi$$

The results from the longitude and latitude rotations are

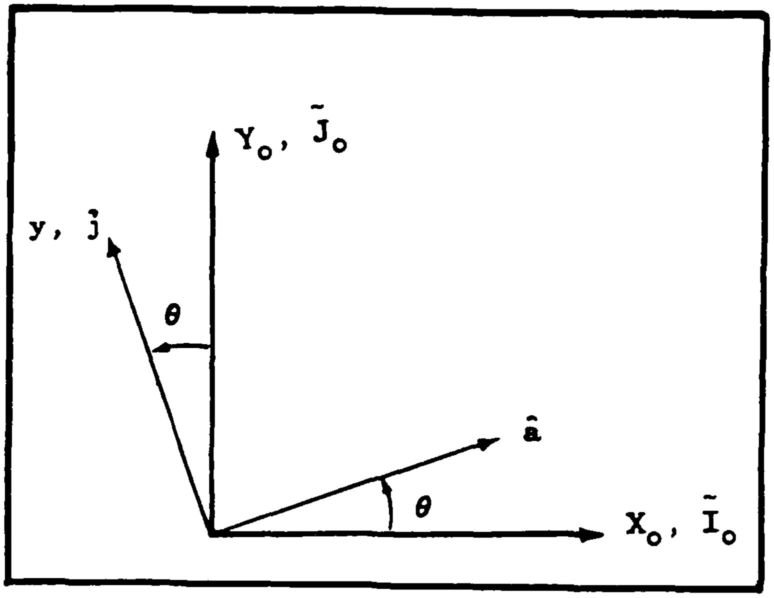

Figure 5. Longitude Rotation

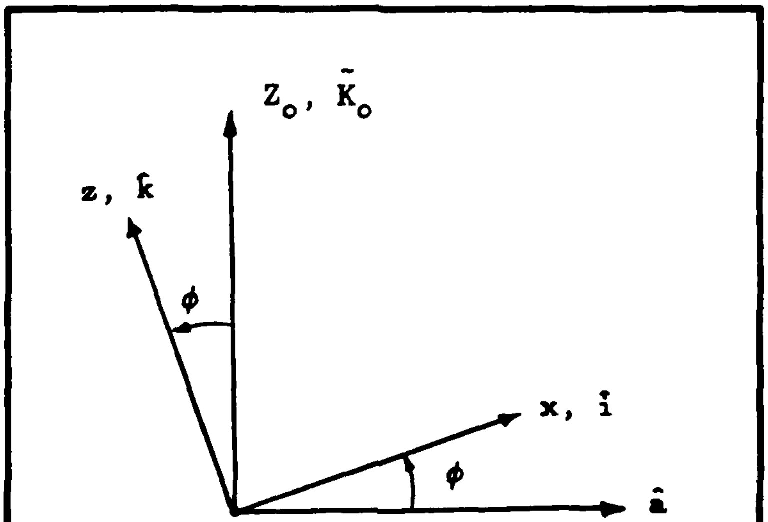

Figure 6. Latitude Rotation

combined and presented below:

$$\begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix} = \begin{bmatrix} \cos\phi\cos\theta & \cos\phi\sin\theta & \sin\phi \\ -\sin\theta & \cos\theta & 0 \\ -\cos\theta\sin\phi & -\sin\theta\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \tilde{I}_o \\ \tilde{J}_o \\ K_o \end{bmatrix} \tag{2.3}$$

and

$$\begin{bmatrix} \tilde{I}_o \\ \tilde{J}_o \\ K_o \end{bmatrix} = \begin{bmatrix} \cos\phi\cos\theta & -\sin\theta & \sin\phi\cos\theta \\ -\sin\theta\cos\phi & \cos\theta & \sin\phi\sin\theta \\ -\sin\phi & 0 & \cos\phi \end{bmatrix} \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix} \tag{2.4}$$

At this point enough information is known to make coordinate transformations between the XYZ inertial frame and the rotating xyz vehicle point mass frame. However, the transformation between the xyz and the $x_o y_o z_o$ frames is required. Figure 4 presented the vehicle centered coordinate systems that are used in the equations of motion. $\hat{b}$ is a unit vector that is used to make the intermediate coordinate transformations easier to follow.

<u>4th Rotation, Heading Angle</u>. The fourth rotation is a heading angle rotation about the x axis. From Figure 7

$$\hat{b} = \hat{j}\cos\psi + \hat{k}\sin\psi \qquad \text{and} \qquad \hat{k}_o = -\hat{j}\sin\psi + \hat{k}\cos\psi$$

<u>5th Rotation, Flight Path Angle</u>. The fifth rotation is a flight path angle rotation about the z_o axis. It can be seen from Figure 8 that

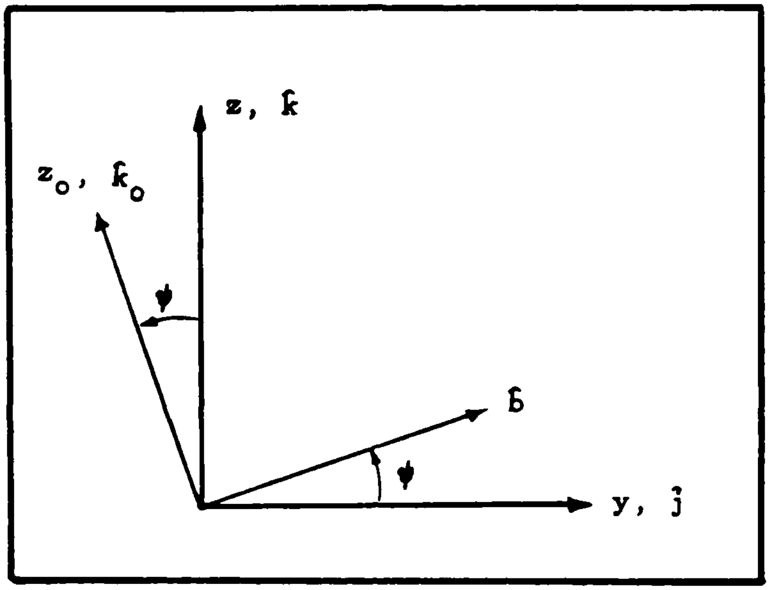

Figure 7. Heading Angle Rotation

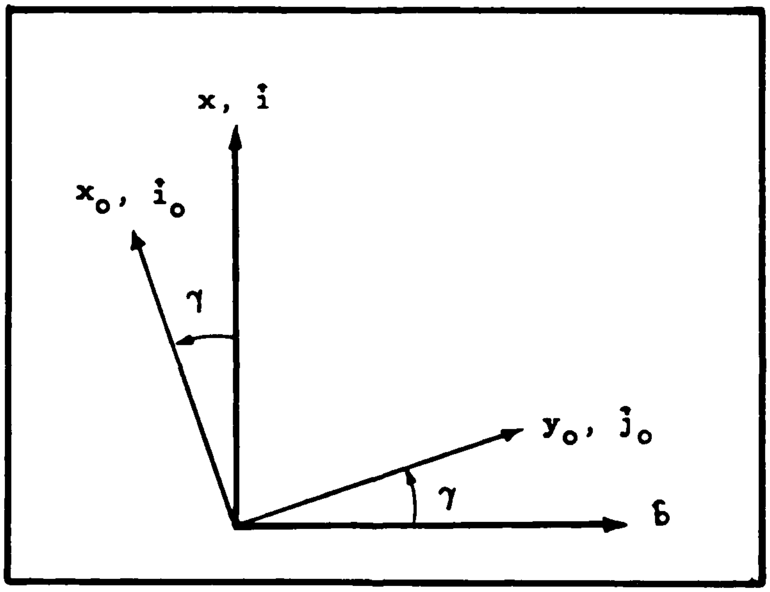

Figure 8. Flight Path Angle Rotation

$$\hat{i}_o = \hat{i}\cos\gamma - \hat{k}\sin\gamma \qquad \text{and} \qquad \hat{j}_o = \hat{i}\sin\gamma + \hat{k}\cos\gamma$$

The results from the heading angle and flight path angle rotations are combined and presented below:

$$\begin{bmatrix} \hat{i}_o \\ \hat{j}_o \\ \hat{k}_o \end{bmatrix} = \begin{bmatrix} \cos\gamma & -\sin\gamma\cos\psi & -\sin\gamma\sin\psi \\ \sin\gamma & \cos\gamma\cos\psi & \cos\gamma\sin\psi \\ 0 & -\sin\psi & \cos\psi \end{bmatrix} \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix} \tag{2.5}$$

and

$$\begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix} = \begin{bmatrix} \cos\phi\cos\theta & -\sin\gamma & 0 \\ \sin\gamma\cos\psi & \cos\gamma\cos\psi & \sin\psi \\ \sin\gamma\sin\psi & -\cos\gamma\sin\psi & \cos\psi \end{bmatrix} \begin{bmatrix} \hat{i}_o \\ \hat{j}_o \\ \hat{k}_o \end{bmatrix} \tag{2.6}$$

Combining all five rotations gives the relations for a complete coordinate transformation between the $x_o y_o z_o$ and XYZ systems.

$$\begin{bmatrix} \hat{i}_o \\ \hat{j}_o \\ \hat{k}_o \end{bmatrix} = \begin{bmatrix} \cos\gamma & -\sin\gamma\cos\psi & -\sin\gamma\sin\psi \\ \sin\gamma & \cos\gamma\cos\psi & \cos\gamma\sin\psi \\ 0 & -\sin\psi & \cos\psi \end{bmatrix} \begin{bmatrix} \cos\phi\cos\theta & \cos\phi\sin\theta & \sin\phi \\ -\sin\theta & \cos\theta & 0 \\ -\cos\theta\sin\phi & -\sin\theta\sin\phi & \cos\phi \end{bmatrix} \cdot$$

$$\cdot \begin{bmatrix} \cos\omega t & \sin\omega t & 0 \\ -\sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \tilde{I} \\ \tilde{J} \\ \tilde{K} \end{bmatrix} \tag{2.7}$$

and

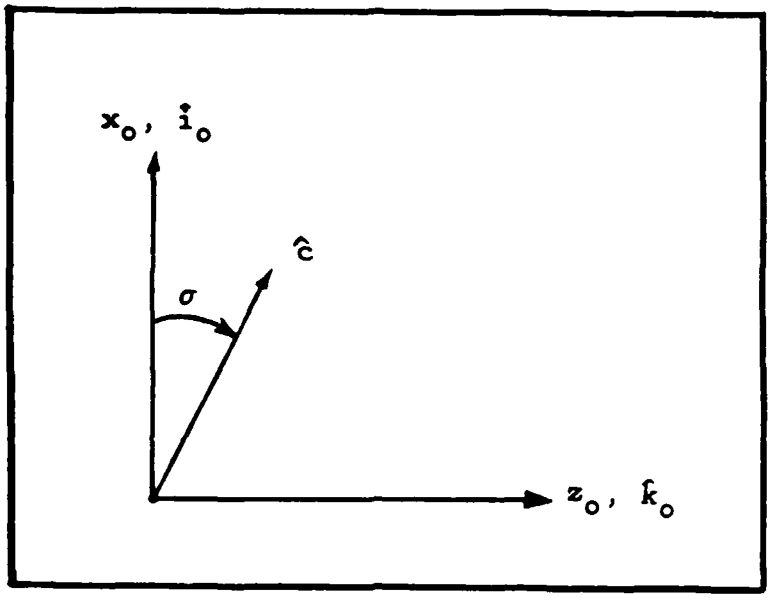

Figure 9. Bank Angle Rotation

$$\begin{bmatrix} \tilde{I} \\ \tilde{J} \\ \tilde{K} \end{bmatrix} = \begin{bmatrix} \cos\omega t & -\sin\omega t & 0 \\ \sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\phi\cos\theta & -\sin\theta & \sin\phi\cos\theta \\ -\sin\theta\cos\phi & \cos\theta & \sin\phi\sin\theta \\ -\sin\phi & 0 & \cos\phi \end{bmatrix} \cdot$$

$$\cdot \begin{bmatrix} \cos\phi\cos\theta & -\sin\gamma & 0 \\ \sin\gamma\cos\psi & \cos\gamma\cos\psi & \sin\psi \\ \sin\gamma\sin\psi & -\cos\gamma\sin\psi & \cos\psi \end{bmatrix} \cdot \begin{bmatrix} \hat{i}_o \\ \hat{j}_o \\ \hat{k}_o \end{bmatrix} \tag{2.8}$$

The only other coordinate transformation relation that requires consideration at this point is the rotation to allow for banked flight. Defining σ as bank angle in Figure 9 gives the following relation:

$$\hat{c} = \hat{i}_o\cos\sigma + \hat{k}_o\sin\sigma \tag{2.9}$$

where $\hat{c}$ is a unit vector in the direction of the lift force.

From Figure 6 it can be seen that aerodynamic lift acts in the $\hat{c}$ direction, drag acts in the $-\hat{j}$ direction and the vehicle's velocity vector points in the $+\hat{j}$ direction.

Hence, lift is given by

$$\vec{L} = L\hat{c} = (L\cos\sigma)\hat{i}_o + (L\sin\sigma)\hat{k}_o \tag{2.13}$$

From Eqs (2.9) and (2.5)

$$\vec{L} = L\cos\sigma\left[\cos\gamma\hat{i} - \sin\gamma\cos\psi\hat{j} - \sin\gamma\sin\psi\hat{k}\right] + L\sin\sigma\left[-\sin\psi\hat{j} + \cos\psi\hat{k}\right]$$

$$\vec{L} = L(\cos\sigma\cos\gamma)\hat{i} - L(\cos\sigma\sin\gamma\cos\psi + \sin\sigma\sin\psi)\hat{j} + L(\sin\sigma\cos\psi - \cos\sigma\sin\gamma\sin\psi)\hat{k} \tag{2.14}$$

The drag components are found in terms of the vehicle centered unit vectors by application of Eq (2.6):

$$\vec{D} = -D\hat{j}_o = -D(\sin\gamma)\hat{i} - D(\cos\gamma\cos\psi)\hat{j} - D(\cos\gamma\sin\psi)\hat{k} \tag{2.15}$$

The vehicle's local or relative velocity with respect to its own reference frame is

$$\vec{V}\Big|_{\hat{i}\hat{j}\hat{k}} = V(\sin\gamma)\hat{i} + V(\cos\gamma\cos\psi)\hat{j} + V(\cos\gamma\sin\psi)\hat{k} \tag{2.16}$$

The planet's rotation needs to be accounted for to obtain the vehicle's inertial velocity from Eqn (2.16). A velocity component due to the planet's rotation is added to the vehicle's local velocity to form the inertial vehicle velocity. The planet's rotation velocity component is a function of latitude; at the equator this velocity component has its maximum value and at the poles it is zero.

Vehicle's Inertial Velocity = $\vec{V}\Big|_{\tilde{I}\tilde{J}\tilde{K}}$

Velocity component due to planet rotation = $\vec{V}_r$

Vehicle's local velocity = $\vec{V}\Big|_{\hat{i}\hat{j}\hat{k}}$

$$\vec{V}\Big|_{\tilde{I}\tilde{J}\tilde{K}} = \vec{V}\Big|_{\hat{i}\hat{j}\hat{k}} + \vec{V}_r \tag{2.17}$$

$$\vec{V}_r = V_r\hat{j} = \omega r\cos\phi\hat{j} \tag{2.18}$$

Combining Eqs (2.16), (2.17), and (2.18) gives an expression for the inertial velocity of the flight vehicle.

$$\vec{V}\Big|_{\tilde{I}\tilde{J}\tilde{K}} = V(\sin\gamma)\hat{i} + V(\cos\gamma\cos\psi + \omega r\cos\phi)\hat{j}$$

$$+ V(\cos\gamma\sin\psi)\hat{k} \tag{2.19}$$

Another expression for inertial velocity can be derived from Eq (2.10) and compared to Eq (2.19) to produce three of the equations of motion.

$$\vec{V}\Big|_{\tilde{I}\tilde{J}\tilde{K}} = \frac{d\vec{r}}{dt}\Big|_{\tilde{I}\tilde{J}\tilde{K}} = \frac{d\vec{r}}{dt}\Big|_{\hat{i}\hat{j}\hat{k}} + \vec{\Omega} \times \vec{r}\Big|_{\hat{i}\hat{j}\hat{k}}$$

In this equation $\vec{\Omega}$ is the vector sum of two rotation rates between three coordinate systems, the XYZ, $X_0Y_0Z_0$, and xyz systems. Therefore, Ω is the sum of the angular velocity of the rotating planet frame about the inertial frame and the angular velocity of the vehicle centered reference frame about the rotating planet frame. This can be expressed by

$$\vec{\Omega} = \vec{\Omega}\Big|_{XYZ \rightarrow X_0 Y_0 Z_0} + \vec{\Omega}\Big|_{X_0 Y_0 Z_0 \rightarrow xyz}$$

From Figure 1

$$\vec{\Omega} = \vec{\omega} + \dot{\vec{\theta}} + \dot{\vec{\phi}} = \left[\omega + \frac{d\theta}{dt}\right]\tilde{K}_0 - \frac{d\phi}{dt}\hat{j}$$

and with Eqn (2.4)

$$\vec{\Omega} = \left[\omega + \frac{d\theta}{dt}\right]\sin\phi\hat{i} - \frac{d\phi}{dt}\hat{j} + \left[\omega + \frac{d\theta}{dt}\right]\cos\phi\hat{k} \tag{2.20}$$

Since $\vec{r}\Big|_{\hat{i}\hat{j}\hat{k}} = r\hat{i}$

$$\frac{d\vec{r}}{dt}\Big|_{\hat{i}\hat{j}\hat{k}} = \frac{dr}{dt}\hat{i} + r\frac{d\hat{i}}{dt}\Big|_{\hat{i}\hat{j}\hat{k}} = \frac{dr}{dt}\hat{i} \tag{2.21}$$

and $\vec{\Omega} \times \vec{r}\Big|_{\hat{i}\hat{j}\hat{k}} = r\left[\omega + \frac{d\theta}{dt}\right]\cos\phi\hat{j} + \frac{d\phi}{dt}\hat{k}$

Hence, the second equation for inertial velocity in terms of the vehicle centered reference frame coordinates is

$$\vec{V}\Big|_{\tilde{I}\tilde{J}\tilde{K}} = \frac{dr}{dt}\hat{i} + r\cos\phi(\omega + \frac{d\theta}{dt})\hat{j} + r\frac{d\phi}{dt}\hat{k} \tag{2.22}$$

Three kinematic equations of motion are derived by equating the two inertial velocity expressions, Eqs (2.19) and (2.22), and comparing like terms.

$$\frac{d\vec{r}}{dt}\hat{i} + r\cos\phi(\omega + \frac{d\theta}{dt})\hat{j} + r\frac{d\phi}{dt}\hat{k} = V(\sin\gamma)\hat{i} +$$

$$+ V(\cos\gamma\cos\psi + \omega r\cos\phi)\hat{j} + V(\cos\gamma\sin\psi)\hat{k}$$

Kinematic Equations of Motion.

$$\hat{i} \text{ terms:} \quad \frac{dr}{dt} = V\sin\gamma \tag{2.23}$$

$$\hat{j} \text{ terms:} \quad \frac{d\theta}{dt} = \frac{V\cos\gamma\cos\psi}{r\cos\phi} \tag{2.24}$$

$$\hat{k} \text{ terms:} \quad \frac{d\phi}{dt} = \frac{V\cos\gamma\sin\psi}{r} \tag{2.25}$$

Derivation of the Force Equations of Motion

To derive additional equations of motion, the inertial acceleration is calculated. From Eq (2.12)

$$\vec{a}\Big|_{\overline{IJK}} = \vec{a}\Big|_{\hat{i}\hat{j}\hat{k}} + 2\vec{\Omega} \times \dot{\vec{r}}\Big|_{\hat{i}\hat{j}\hat{k}} + \dot{\vec{\Omega}} \times \vec{r}\Big|_{\hat{i}\hat{j}\hat{k}} + \vec{\Omega} \times \left[\vec{\Omega} \times \vec{r}\Big|_{\hat{i}\hat{j}\hat{k}}\right]$$

Each of the four terms in the above equation is derived separably for clarity.

1st Acceleration Term. $\vec{a}\Big|_{\hat{i}\hat{j}\hat{k}} = \frac{d\vec{r}^2}{dt^2}\Big|_{\hat{i}\hat{j}\hat{k}}$

From Eqs (2.21) and (2.23)

$$\dot{\vec{r}}\Big|_{\hat{i}\hat{j}\hat{k}} = \frac{dr}{dt}\hat{i} = V\sin\gamma\hat{i}$$

Therefore $\vec{a}\Big|_{\hat{i}\hat{j}\hat{k}} = \left[\sin\gamma\frac{dV}{dt} + V\cos\gamma\frac{d\gamma}{dt}\right]\hat{i}$ (2.26)

<u>2nd Acceleration Term</u>. $2\vec{\Omega} \times \dot{\vec{r}}\Big|_{\hat{i}\hat{j}\hat{k}}$

From Eqs (2.20)

$$\vec{\Omega} = \left[\omega + \frac{d\theta}{dt}\right]\sin\phi\hat{i} - \frac{d\phi}{dt}\hat{j} + \left[\omega + \frac{d\theta}{dt}\right]\cos\phi\hat{k}$$

and from above

$$\dot{\vec{r}}\Big|_{\hat{i}\hat{j}\hat{k}} = \frac{dr}{dt}\hat{i} = V\sin\gamma\hat{i}$$

Taking twice the value of the cross product of these two equations gives the second inertial acceleration term.

$$2\vec{\Omega} \times \dot{\vec{r}}\Big|_{\hat{i}\hat{j}\hat{k}} = \left[2V\sin\gamma\cos\phi\left(\omega + \frac{d\theta}{dt}\right)\right]\hat{j} + \left[2V\sin\gamma\frac{d\phi}{dt}\right]\hat{k} \qquad (2.27)$$

<u>3rd Acceleration Term</u>. $\dot{\vec{\Omega}} \times \vec{r}\Big|_{\hat{i}\hat{j}\hat{k}}$

Taking the time derivative of Eq (2.20) gives

$$\dot{\vec{\Omega}} = \left[\omega + \frac{d\theta}{dt}\right]\cos\phi\frac{d\phi}{dt}\hat{i} + \frac{d^2\theta}{dt^2}\sin\phi\hat{i} + \left[\omega + \frac{d\theta}{dt}\right]\sin\phi\frac{d\hat{i}}{dt} - \frac{d^2\phi}{dt^2}\hat{j}$$

$$- \left[\omega + \frac{d\theta}{dt}\right]\sin\phi\frac{d\phi}{dt}\hat{k} + \frac{d^2\theta}{dt^2}\cos\phi\hat{k} + \left[\omega + \frac{d\theta}{dt}\right]\cos\phi\frac{d\hat{k}}{dt}$$

Rewritten

$$\dot{\vec{\Omega}} = \left[\left[\omega + \frac{d\theta}{dt}\right]\cos\phi\frac{d\phi}{dt} + \frac{d^2\theta}{dt^2}\sin\phi\right]\hat{i} - \frac{d^2\phi}{dt^2}\hat{j}$$

$$+ \left[\frac{d^2\theta}{dt^2}\cos\phi - \left[\omega + \frac{d\theta}{dt}\right]\sin\phi\frac{d\phi}{dt}\right]\hat{k} \qquad (2.28)$$

Therefore, the third term of Eq (2.12) is

$$\dot{\vec{\Omega}} \times \vec{r}\Big|_{\hat{i}\hat{j}\hat{k}} = \left[r\frac{d^2\theta}{dt^2}\cos\phi - \left[\omega + \frac{d\theta}{dt}\right]r\sin\phi\frac{d\phi}{dt}\right]\hat{j} + r\frac{d^2\phi}{dt^2}\hat{k} \qquad (2.29)$$

<u>4th Acceleration Term</u>. $\vec{\Omega} \times \left[\vec{\Omega} \times \vec{r}\Big|_{\hat{i}\hat{j}\hat{k}}\right]$

Taking the cross product of Eqs (2.20) and (2.21) gives

$$\vec{\Omega} \times \vec{r}\Big|_{\hat{i}\hat{j}\hat{k}} = r\left[\omega + \frac{d\theta}{dt}\right]\cos\phi\hat{j} + r\frac{d\phi}{dt}\hat{k}$$

Taking another cross product produces the fourth inertial acceleration term.

$$\vec{\Omega} \times (\vec{\Omega} \times \vec{r}\Big|_{\hat{i}\hat{j}\hat{k}}) = -\left[r\left[\omega + \frac{d\theta}{dt}\right]^2\cos^2\phi + r\frac{d^2\phi}{dt^2}\right]\hat{i}$$

$$- \left[r\frac{d\phi}{dt}\left[\omega + \frac{d\theta}{dt}\right]\sin\phi\right]\hat{j} + \left[r\left[\omega + \frac{d\theta}{dt}\right]^2\sin\phi\cos\phi\right]\hat{k}$$

<u>Total Inertial Acceleration</u>. Combining Eqs (2.26), (2.27), (2.29), and (2.30) gives the total inertial acceleration in terms of the vehicle centered direction unit vectors.

$$\vec{a}\Big|_{\tilde{I}\tilde{J}\tilde{K}} = \left[\sin\gamma\frac{dV}{dt} + V\cos\gamma\frac{d\gamma}{dt} - r\left[\omega + \frac{d\theta}{dt}\right]^2\cos^2\phi - r\frac{d^2\phi}{dt^2}\right]\hat{i}$$

$$+ \left[2V\sin\gamma\cos\phi\left[\omega + \frac{d\theta}{dt}\right] + r\frac{d^2\theta}{dt^2}\cos\phi - 2\left[\omega + \frac{d\theta}{dt}\right]r\sin\phi\frac{d\phi}{dt}\right]\hat{j}$$

$$+ \left[r\left[\omega + \frac{d\theta}{dt}\right]^2 \sin\phi\cos\phi + r\frac{d^2\phi}{dt^2} + 2V\sin\gamma\frac{d\phi}{dt}\right]\hat{k} \qquad (2.31)$$

This equation can be simplified by substitution of the three kinematic equations of motion that were previously derived, Eqs (2.23), (2.24), and (2.25). Because of the size of this new equation, each direction component of the inertial acceleration, Eq (2.31), is examined separably below.

<u>$\hat{i}$th Inertial Acceleration Component</u>. The $\hat{i}$th term of Eq (2.31) is

$$\hat{i}\text{th} = \left[\sin\gamma\frac{dV}{dt} + V\cos\gamma\frac{d\gamma}{dt} - r\left[\omega + \frac{d\theta}{dt}\right]^2 \cos^2\phi - r\frac{d^2\phi}{dt^2}\right]$$

Substituting in for $\frac{d\theta}{dt}$ and $\frac{d\phi}{dt}$ with Eqs (2.24) and (2.25)

$$\hat{i}\text{th} = \sin\gamma\frac{dV}{dt} + V\cos\gamma\frac{d\gamma}{dt} - r\omega^2\cos^2\phi - 2\omega r\left[\frac{V\cos\gamma\cos\psi}{r\cos\phi}\right]\cos^2\phi$$

$$- r\cos^2\phi\left[\frac{V^2\cos^2\gamma\cos^2\psi}{r^2\cos^2\phi}\right] - r\left[\frac{V^2\cos^2\gamma\sin^2\psi}{r^2}\right]$$

This can be simplified to

$$\hat{i}\text{th} = \sin\gamma\frac{dV}{dt} + V\cos\gamma\frac{d\gamma}{dt} - \frac{V^2}{r}\cdot\cos^2\phi - 2\omega V\cos\gamma\cos\psi\cos\phi$$

$$- \omega^2 r\cos^2\phi \qquad (2.32)$$

$\hat{j}$th Inertial Acceleration Component. The $\hat{j}$th term of Eq (2.31) is

$$\hat{j}\text{th} = \left[2V\sin\gamma\cos\phi\left[\omega + \frac{d\theta}{dt}\right] + r\frac{d^2\theta}{dt^2}\cos\phi - 2\left[\omega + \frac{d\theta}{dt}\right]r\sin\phi\frac{d\phi}{dt}\right]$$

Substituting in for $\frac{d\theta}{dt}$ and $\frac{d\phi}{dt}$ produces

$$\hat{j}\text{th} = 2\omega V\sin\gamma\cos\phi + 2V\sin\gamma\cos\phi\left[\frac{V\cos\gamma\cos\psi}{r\cos\phi}\right]$$

$$+ r\cos\phi\frac{d}{dt}\left[\frac{V\cos\gamma\cos\psi}{r\cos\phi}\right] - 2\omega r\sin\phi\left[\frac{V\cos\gamma\sin\psi}{r}\right]$$

$$- 2r\sin\phi\left[\frac{V\cos\gamma\sin\psi}{r}\right]\left[\frac{V\cos\gamma\cos\psi}{r\cos\phi}\right]$$

This equation simplifies to the following expression:

$$\hat{j}\text{th} = \frac{d}{dt}\left[\frac{V\cos\gamma\cos\psi}{r\cos\phi}\right]r\cos\phi + 2\omega V\sin\gamma\cos\phi + 2\frac{V^2}{r}\cdot\sin\gamma\cos\phi\cos\psi$$

$$- 2\omega V\sin\phi\cos\gamma\sin\psi - 2\frac{V^2}{r}\cdot\tan\phi\cos^2\gamma\sin\psi\cos\psi$$

The first term of this equation is

$$\frac{d}{dt}\left[\frac{V\cos\gamma\cos\psi}{r\cos\phi}\right]r\cos\phi = \left[\cos\gamma\cos\psi\frac{dV}{dt} - V\sin\gamma\cos\psi\frac{d\gamma}{dt}\right.$$

$$\left. - V\cos\gamma\sin\psi\frac{d\psi}{dt}\right] - \frac{V\cos\gamma\cos\psi}{r^2\cos^2\phi}\left[\cos\phi\frac{dr}{dt} - r\sin\phi\frac{d\phi}{dt}\right]$$

Simplifying and substituting in Eqs (2.23) and (2.25) gives

$$\frac{d}{dt}\left[\frac{V\cos\gamma\cos\psi}{r\cos\phi}\right]r\cos\phi = \cos\gamma\cos\psi\frac{dV}{dt} - V\sin\gamma\cos\psi\frac{d\gamma}{dt} - V\cos\gamma\sin\psi\frac{d\psi}{dt}$$

$$- \frac{V^2}{r} \cdot \sin\gamma\cos\gamma\cos\psi + \frac{V^2}{r} \cdot \cos^2\gamma\cos\psi\sin\psi\tan\phi$$

Therefore, the entire $\hat{j}$th term of Eq (2.31) is

$$\hat{j}\text{th} = \cos\gamma\cos\psi\frac{dV}{dt} - V\sin\gamma\cos\psi\frac{d\gamma}{dt} - V\cos\gamma\sin\psi\frac{d\psi}{dt}$$

$$+ 2\omega V\left[\sin\gamma\cos\phi - \sin\phi\cos\gamma\sin\psi\right]$$

$$+ \frac{V^2}{r} \cdot \cos\gamma\cos\psi\left[\sin\gamma - \cos\gamma\sin\psi\tan\phi\right] \qquad (2.33)$$

<u>$\hat{k}$th Inertial Acceleration Component</u>. The $\hat{k}$th term of Eq (2.31) is

$$\hat{k}\text{th} = r\frac{d^2\phi}{dt^2} + 2V\sin\gamma\frac{d\phi}{dt} + r\left[\omega + \frac{d\theta}{dt}\right]^2 \sin\phi\cos\phi$$

The first term in this $\hat{k}$th component requires some manipulation. Differentiating Eq (2.25) and multiplying by r gives

$$r\frac{d^2\phi}{dt^2} = r \cdot \frac{d}{dt}\left[\frac{V\cos\gamma\sin\psi}{r}\right] = \cos\gamma\sin\psi\frac{dV}{dt} - V\sin\gamma\sin\psi\frac{d\gamma}{dt}$$

$$+ V\cos\gamma\cos\psi\frac{d\psi}{dt} - \frac{V^2}{r} \cdot \sin\gamma\cos\gamma\sin\psi \qquad (2.34)$$

Substituting Eqs (2.34), (2.23), and (2.24) into the $\hat{k}$th term produces

$$\hat{k}\text{th} = 2\frac{V^2}{r} \cdot \sin\gamma\cos\gamma\sin\psi + \omega^2 r\sin\phi\cos\phi + \frac{V^2\cos^2\gamma\cos^2\psi\sin\phi}{r\cos\phi}$$

$$+ 2\omega V\cos\gamma\cos\psi\sin\phi + \cos\gamma\sin\psi\frac{dV}{dt} - V\sin\gamma\sin\psi\frac{d\gamma}{dt}$$

$$+ V\cos\gamma\cos\psi\frac{d\psi}{dt} - \frac{V^2}{r}\cdot\sin\gamma\cos\gamma\sin\psi$$

This equation can be simplified to the following expression:

$$\hat{k}\text{th} = \frac{V^2}{r}\cdot\cos\gamma(\cos\gamma\cos^2\psi\tan\phi + \sin\gamma\sin\psi) + \cos\gamma\sin\psi\frac{dV}{dt}$$
$$- V\sin\gamma\sin\psi\frac{d\gamma}{dt} + V\cos\gamma\cos\psi\frac{d\psi}{dt} + 2\omega V\cos\gamma\cos\psi\sin\phi$$
$$+ \omega^2 r\sin\phi\cos\phi \qquad (2.35)$$

<u>Total Inertial Acceleration</u>. Adding together the $\hat{i}$th, $\hat{j}$th, and $\hat{k}$th acceleration terms given by Eqs (2.32), (2.33), and (2.35) gives an expression for the total inertial acceleration on the vehicle.

$$\vec{a}\Big|_{\overline{IJK}} = \Big[\sin\gamma\frac{dV}{dt} + V\cos\gamma\frac{d\gamma}{dt} - 2\omega V\cos\gamma\cos\psi\cos\phi - \frac{V^2}{r}\cdot\cos^2\gamma$$
$$- \omega^2 r\cos^2\phi\Big]\hat{i} + \Big[\cos\gamma\cos\psi\frac{dV}{dt} - V\sin\gamma\cos\psi\frac{d\gamma}{dt} - V\cos\gamma\sin\psi\frac{d\psi}{dt}$$
$$+ 2V\omega(\cos\phi\sin\gamma - \sin\phi\cos\gamma\sin\psi)$$
$$+ \frac{V^2}{r}\cdot\cos\gamma\cos\psi(\sin\gamma - \sin\psi\cos\gamma\tan\phi)\Big]\hat{j}$$
$$+ \Big[\cos\gamma\sin\psi\frac{dV}{dt} + V\cos\gamma\cos\psi\frac{d\psi}{dt} \qquad (2.36)$$
$$- V\sin\gamma\sin\psi\frac{d\gamma}{dt} + 2\omega V\sin\phi\cos\gamma\cos\psi$$
$$+ \omega^2 r\sin\phi\cos\phi + \frac{V^2}{r}\cdot\cos\gamma(\sin\gamma\sin\psi + \cos^2\psi\tan\phi\cos\gamma)\Big]\hat{k}$$

Another equation for inertial acceleration can be derived by examining the forces on the vehicle. It is assumed that the only forces acting on the vehicle are gravity and aerodynamic lift and drag.

Therefore $$\vec{a}\Big|_{\widetilde{IJK}} = \sum \frac{\vec{F}}{m} \tag{2.37}$$

and $$\vec{a}\Big|_{\widetilde{IJK}} = \frac{d\vec{V}}{dt}\Big|_{\widetilde{IJK}} = \frac{1}{m}(\vec{L} + \vec{D}) + \vec{g} \tag{2.38}$$

where the gravitational force, g, is a function of r and acts in the negative radial direction.

$$\vec{g} = -g(r)\hat{i} \tag{2.39}$$

Lift and drag are given by Eqs (2.14) and (2.15):

$$\vec{L} = (L\cos\sigma\cos\gamma)\hat{i} - (L\cos\sigma\sin\gamma\cos\psi + L\sin\sigma\sin\psi)\hat{j}$$
$$- (L\cos\sigma\sin\gamma\sin\psi - L\sin\sigma\cos\psi)\hat{k}$$

$$\vec{D} = (-D\sin\gamma)\hat{i} - (D\cos\gamma\cos\psi)\hat{j} - (D\cos\gamma\sin\psi)\hat{k}$$

With substitution, a second equation is found for the total inertial acceleration of the vehicle:

$$\vec{a}\Big|_{\widetilde{IJK}} = \left[\frac{L}{m}\cos\sigma\cos\gamma - \frac{D}{m}\sin\gamma - g\right]\hat{i}$$
$$- \left[\frac{L}{m}\cos\sigma\sin\gamma\cos\psi + \frac{L}{m}\sin\sigma\sin\psi + \frac{D}{m}\cos\gamma\cos\psi\right]\hat{j}$$
$$- \left[\frac{L}{m}\cos\sigma\sin\gamma\sin\psi - \frac{L}{m}\sin\sigma\cos\psi + \frac{D}{m}\cos\gamma\sin\psi\right]\hat{k} \tag{2.40}$$

By setting the two different equations for inertial acceleration equal and comparing like terms, three additional equations of motion can be derived. Equating Eqs (2.40) and (2.35) produces the following expressions:

$\hat{i}$ Direction Term.

$$\sin\gamma\frac{dV}{dt} + V\cos\gamma\frac{d\gamma}{dt} - 2\omega V\cos\gamma\cos\psi\cos\phi - \frac{V^2}{r}\cdot\cos^2\gamma - \omega^2 r\cos^2\phi$$

$$= \frac{1}{m}(L\cos\sigma\cos\gamma - D\sin\gamma) - g \qquad (2.41)$$

$\hat{j}$ Direction Term.

$$\cos\gamma\frac{dV}{dt} - V\sin\gamma\frac{d\gamma}{dt} - V\cos\gamma\tan\psi\frac{d\psi}{dt}$$

$$+ \frac{2V\omega}{\cos\psi}(\cos\phi\sin\gamma-\sin\phi\cos\gamma\sin\psi) + \frac{V^2}{r}\cdot\cos\gamma(\sin\gamma-\sin\psi\cos\gamma\tan\phi)$$

$$= \frac{-1}{m}(L\cos\sigma\sin\gamma + L\sin\sigma\tan\psi + D\cos\gamma)\hat{j} \qquad (2.42)$$

$\hat{k}$ Direction Term.

$$\cos\gamma\frac{dV}{dt} + V\frac{\cos\gamma}{\tan\psi}\frac{d\psi}{dt} - V\sin\gamma\frac{d\gamma}{dt} + 2\omega V\frac{\sin\phi\cos\gamma}{\tan\psi} + \omega^2 r\frac{\sin\phi\cos\phi}{\sin\psi}$$

$$+ \frac{V^2}{r}\cdot\cos\gamma\left[\sin\gamma + \frac{\cos\psi\tan\phi\cos\gamma}{\tan\psi}\right]$$

$$= -\frac{1}{m}(L\cos\sigma\sin\gamma - L\frac{\sin\sigma}{\tan\psi} + D\cos\gamma) \qquad (2.43)$$

These three coupled equations can be reduced by some manipulation. Multiplying Eq (2.43) by -1 and adding the product to Eq (2.42) gives

$$- V\cos\gamma\tan\psi\frac{d\psi}{dt} + \frac{2V\omega}{\cos\psi}(\cos\phi\sin\gamma - \sin\phi\cos\gamma\sin\psi)$$

$$- \frac{V^2}{r}\cdot\cos\gamma\left[\sin\psi\cos\gamma\tan\phi\right] - V\frac{\cos\gamma}{\tan\psi}\frac{d\psi}{dt} - 2\omega V\frac{\sin\phi\cos\gamma}{\tan\psi}$$

$$- \omega^2 r\frac{\sin\phi\cos\phi}{\sin\psi} - \frac{V^2}{r}\cdot\cos\gamma\left[\frac{\cos\psi\tan\phi\cos\gamma}{\tan\psi}\right]$$

$$= - \frac{L}{m}\sin\sigma\tan\psi - \frac{L}{m}\cdot\frac{\sin\sigma}{\tan\psi}$$

By combining terms and noting the trigonometric identities

$$1 + \tan^2\psi = (\cos^2\psi)^{-1} \quad \text{and} \quad \sin^2\psi + \cos^2\psi = 1$$

the above expression can be rewritten to form a new equation of motion:

$$V\frac{d\psi}{dt} = \frac{L}{m}\cdot\frac{\sin\sigma}{\cos\gamma} + 2V\omega(\cos\phi\tan\gamma\sin\psi - \sin\phi)$$

$$- \omega^2 r\left[\frac{\sin\phi\cos\phi\cos\psi}{\cos\gamma}\right] - \frac{V^2}{r}\cdot\cos\gamma\cos\psi\tan\phi \tag{2.44}$$

Substituting Eq (2.44) back into Eq (2.43) produces

$$0 = \cos\gamma\frac{dV}{dt} + 2V\omega\cos\phi\sin\gamma\cos\psi - V\sin\gamma\frac{d\gamma}{dt} + \frac{V^2}{r}\cdot\cos\gamma\sin\gamma$$

$$+ \frac{L}{m}\cos\sigma\sin\gamma + \frac{D}{m}\cos\gamma + \omega^2 r\sin\phi\cos\phi\sin\psi \tag{2.45}$$

Multiplying Eq (2.41) by $-\frac{\cos\gamma}{\sin\gamma}$ and adding this product to Eq (2.45) gives another equation of motion:

$$0 = -V\frac{d\gamma}{dt}\left[\sin\gamma + \frac{\cos^2\gamma}{\sin\gamma}\right] + \frac{V^2}{r}\cdot\cos\gamma\left[\frac{\cos^2\gamma}{\sin\gamma} + \sin\gamma\right]$$

$$+ \frac{L}{m}\cos\sigma\left[\sin\gamma + \frac{\cos^2\gamma}{\sin\gamma}\right] - g\frac{\cos\gamma}{\sin\gamma} + 2V\omega\cos\phi\cos\psi\left[\sin\gamma\right.$$

$$+ \frac{\cos^2\gamma}{\sin\gamma}\Big] + \omega^2 r\cos\phi\left[\sin\phi\sin\psi + \cos\phi\frac{\cos\gamma}{\sin\gamma}\right]$$

Rewritten,

$$V\frac{d\gamma}{dt} = \frac{V^2}{r}\cdot\cos\gamma + \frac{L}{m}\cos\sigma - g\cos\gamma + 2V\omega\cos\phi\cos\psi$$

$$+ \omega^2 r\cos\phi\left[\sin\phi\sin\psi\sin\gamma + \cos\phi\cos\gamma\right] \tag{2.46}$$

Substituting Eq (2.46) into Eq (2.42) produces the last equation of motion:

$$0 = \sin\gamma\frac{dV}{dt} + \frac{V^2}{r}\cdot\cos^2\gamma + \frac{L}{m}\cos\gamma\cos\sigma - g\cos^2\gamma + 2V\omega\cos\gamma\cos\phi\cos\psi$$

$$+ \omega^2 r\cos\gamma\cos\phi\left[\cos\phi\cos\gamma - \sin\phi\sin\psi\sin\gamma\right]$$

$$- 2\omega V\cos\gamma\cos\psi\cos\phi - \frac{V^2}{r}\cdot\cos^2\gamma - \omega^2 r\cos^2\phi - \frac{L}{m}\cdot\cos\sigma\cos\gamma$$

$$+ \frac{D}{m}\sin\gamma + g$$

This can be rewritten

$$\frac{dV}{dt} = -\frac{D}{m} - g\sin\gamma + \omega^2 r\cos\phi\left[\cos\phi\sin\gamma - \sin\phi\sin\psi\cos\gamma\right] \tag{2.48}$$

The Equations of Motion

In summary, the following six equations of motion for three-dimensional atmospheric entry for a rotating planet have been derived and are listed below for convenience.

$$\frac{dV}{dt} = -\frac{D}{m} - g\sin\gamma + \omega^2 r\cos\phi\left[\cos\phi\sin\gamma - \sin\phi\sin\psi\cos\gamma\right] \tag{2.49}$$

$$V\frac{d\gamma}{dt} = \frac{V^2}{r}\cdot\cos\gamma + \frac{L}{m}\cdot\cos\sigma - g\cos\gamma + 2V\omega\cos\phi\cos\psi$$

$$+ \omega^2 r\cos\phi\left[\sin\phi\sin\psi\sin\gamma + \cos\phi\cos\gamma\right] \quad (2.50)$$

$$V\frac{d\psi}{dt} = \frac{L}{m}\cdot\frac{\sin\sigma}{\cos\gamma} + 2V\omega\left[\cos\phi\tan\gamma\sin\psi - \sin\phi\right]$$

$$- \omega^2 r\left[\frac{\sin\phi\cos\phi\cos\psi}{\cos\gamma}\right] - \frac{V^2}{r}\cdot\cos\gamma\cos\psi\tan\phi \quad (2.51)$$

$$\frac{dr}{dt} = V\sin\gamma \quad (2.52)$$

$$\frac{d\theta}{dt} = \frac{V\cos\gamma\cos\psi}{r\cos\phi} \quad (2.53)$$

$$\frac{d\phi}{dt} = \frac{V\cos\gamma\sin\psi}{r} \quad (2.54)$$

In the next section, these equations of motion are transformed into a form more convenient to analyze.

III. Approximations and Manipulation of the Equations of Motion

The equations of motion for three-dimensional atmospheric entry for a rotating spherical planet were derived in Section II. In this section, assumptions and approximations used in this thesis are defined and discussed. In addition, the equations of motion are transformed into a form more convenient to examine and solve in later sections. The independent variable is changed from time to non-dimensional altitude, h, and the equations of motion are transformed into non-dimensional form by the introduction of non-dimensional variables. A coordinate system transformation is undertaken to utilize variables which are more convenient for atmospheric entry analysis.

The equations of motion derived in Section II are given by

$$\frac{dV}{dt} = -\frac{D}{m} - g\sin\gamma + \omega^2 r\cos\phi(\cos\phi\sin\gamma - \sin\phi\sin\psi\cos\gamma) \tag{3.1}$$

$$V\frac{d\gamma}{dt} = \frac{V^2}{r}\cdot\cos\gamma + \frac{L}{m}\cos\sigma - g\cos\gamma + 2V\omega\cos\phi\cos\psi + \omega^2 r\cos\phi(\sin\phi\sin\psi\sin\gamma + \cos\phi\cos\gamma) \tag{3.2}$$

$$V\frac{d\psi}{dt} = \frac{L}{m}\cdot\frac{\sin\sigma}{\cos\gamma} - \frac{V^2}{r}\cdot\cos\gamma\cos\psi\tan\phi + 2V\omega(\cos\phi\tan\gamma\sin\psi - \sin\phi) - \frac{\omega^2 r\sin\phi\cos\phi\cos\psi}{\cos\gamma} \tag{3.3}$$

$$\frac{dr}{dt} = V\sin\gamma \tag{3.4}$$

$$\frac{d\theta}{dt} = \frac{V\cos\gamma\cos\psi}{r\cos\phi} \tag{3.5}$$

$$\frac{d\phi}{dt} = \frac{V\cos\gamma\sin\psi}{r} \tag{3.6}$$

Note that the equations of motion for three-dimensional, non-rotating planetary entry can be derived from the rotating equations simply by setting the planet rotation rate, ω, to zero. The equations of motion for the non-rotating planet assumption are therefore

$$\frac{dV}{dt} = -\frac{D}{m} - g\sin\gamma \tag{3.7}$$

$$V\frac{d\gamma}{dt} = \frac{V^2}{r}\cdot\cos\gamma + \frac{L}{m}\cos\sigma - g\cos\gamma \tag{3.8}$$

$$V\frac{d\psi}{dt} = \frac{L}{m}\cdot\frac{\sin\sigma}{\cos\gamma} - \frac{V^2}{r}\cdot\cos\gamma\cos\psi\tan\phi \tag{3.9}$$

$$\frac{dr}{dt} = V\sin\gamma \tag{3.10}$$

$$\frac{d\theta}{dt} = \frac{V\cos\gamma\cos\psi}{r\cos\phi} \tag{3.11}$$

$$\frac{d\phi}{dt} = \frac{V\cos\gamma\sin\psi}{r} \tag{3.12}$$

As expected, comparison of Eqs (3.1) - (3.6) and Eqs (3.7) - (3.12) shows that the equations of motion are significantly more complicated when the earth's rotation is accounted for.

A few more variables require definition at this point. Let y be altitude and r_* be average planetary equatorial radius. Since r is the radius measured from the center of the spherical planet to the flight vehicle, h is defined as the non-dimensional altitude and is given by

$$h = \frac{y}{r_*} \tag{3.13}$$

where

$$r = r_* + y = r_* + hr_* = r_*(1+h) \tag{3.14}$$

Therefore, $$\frac{dr}{dh} = r_* \tag{3.15}$$

and $$\frac{dt}{dh} = \frac{dr/dh}{dr/dt} = \frac{r_*}{V\sin\gamma} \tag{3.16}$$

These relations will be used in the approximations discussed on the following pages.

Assumptions and Approximations

The rotating planet assumption and the errors associated with the non-rotating approximation were discussed in Section I. Further assumptions and approximations are presented below.

Spherical Planet Assumption. The approximation of an oblate planet by a sphere is very common in analytical flight mechanics analyses and is used here. It is a reasonable assumption for planets having small ellipticity such as Earth, Venus, and Mars but may not be as reasonable for planets such as Jupiter and Saturn which have relatively large equatorial bulges and ellipticities a magnitude greater than Earth's (Chapman, 1958:2). For either case, error introduced by the spherical planet approximation is generally small for near equatorial entry trajectories.

<u>Spherical Atmosphere Assumption</u>. The spherical planet approximation leads to the assumption that the atmosphere is spherically symmetric about the planet. In reality, planets are oblate spheroids causing their atmospheres to also have an oblate form. In addition, other significant deviations from the spherical model occur. A diurnal density bulge occurs over part of Earth's sunlit side due to solar heating. Solar storms and fluctuations in a planet's magnetic field can cause significant changes in density for a given altitude (Wiesel, 1986:66-69). However, since these effects generally occur at altitudes where aerodynamic forces are minimal (and inclusion of more sophisticated density models may introduce overcomplication of the entry problem) these effects are generally assumed to be negligable. The approximation of a spherical planetary atmosphere is perhaps the most limiting assumption to the atmospheric model. This approximation is better for the Terrestial planets, with their slow rotation rates, than for the large, outer planets (Duncan, 1962:276).

<u>Gravitational Model</u>. The spherical planet approximation also leads to the assumption of an inverse square gravitational field. This is given by

$$g(h) = g_* \frac{r_*^2}{r^2} = \frac{g_*}{(1+h)^2} \tag{3.17}$$

where g_* is the gravitational acceleration at the planet surface.

Atmospheric Density Model. As previously discussed, the planet's atmosphere is assumed to be spherical and to rotate with the planet with a constant rotation rate, ω. The planet atmosphere is mathematically modelled by an exponential atmosphere with the inverse atmospheric scale height, β. This is a very common atmospheric model that has been successfully utilized in many studies on planetary atmospheric entry. Atmospheric density, ρ, is given by (Chapman, 1959:4)

$$\frac{d\rho}{\rho} = -\beta dr \tag{3.18}$$

It is sometimes assumed that the product of the inverse of the atmospheric scale height and the vehicle's distance from the planet's center is constant for a given atmosphere. With this model

$$\frac{d\rho}{\rho} = -\frac{\beta r}{r} dr \qquad \text{or} \qquad \rho = \rho_* (r/r_*)^{-\beta r} \tag{3.19}$$

where ρ_* is the density at the surface of the planet.

The product βr has been approximated for many of the planet atmospheres. It has values on the order of 1000 for most of the planets; the mean value of βr is approximately

900 and 350 for Earth and Mars, respectively (Vinh and others, 1980:5).

For this study a strictly exponential atmospheric model is employed where the the inverse atmospheric scale height is assumed to be constant. This approximation allows the atmospheric density equation to be written in the form

$$\rho = \rho_* e^{-\beta y} = \rho_* e^{-h/\epsilon} \tag{3.20}$$

where ϵ is a small number given by

$$\epsilon = \frac{1}{\beta r_*} \tag{3.21}$$

Note that β is dependent on what planet is studied; for Earth, the average scale height is about 7.1 kilometers (Vinh and others, 1980:5) or about 23,300 feet.

For planetary atmospheric entry, r is approximately equal to r_*. This is an accurate approximation because the thickness of an atmosphere is generally very small compared to the planet's radius. For example, the upper altitude limit of Earth's sensible atmosphere is often taken to be 350,000 feet. This value is only about 1.7% of the Earth's radius. Approximating r by r_* leads to

$$\frac{1}{\beta r_*} = \frac{1}{\beta r}$$

and hence the values given in the literature for mean planetary βr are considered equivalent to βr_* . ϵ is therefore a very small number (approximately equal to 1/900 for Earth) allowing it to be utilized as a small parameter in the asymptotic expansions of Section V. Earth atmospheric density values were calculated with this model and plotted in Figure 10 with values obtained from the 1976 U.S. Standard Atmosphere (NOAA, 1976:Table IV). Comparison of these results shows that the accuracy of the exponential density model is reasonable for Earth with a constant value for βr of 900.

Aerodynamic Forces and the Ballistic Coefficient. Lift and drag accelerations are given by the following familiar expressions:

$$\frac{L}{m} = \frac{\rho S V^2 C_L}{2m} \quad \text{and} \quad \frac{D}{m} = \frac{\rho S V^2 C_D}{2m} \tag{3.22}$$

where m is the mass, S is the aerodynamic reference area, V is the velocity, C_L is the lift coefficient, and C_D is the coefficient of drag for the flight vehicle.

The non-dimensional ballistic coefficient, B, is defined to help place the equations of motion into a form easier to solve. It is given by

$$B = \frac{\rho_* S C_D}{2m\beta} \tag{3.23}$$

and is specified for each flight vehicle under consideration. The ballistic coefficient is a function of the vehicle's physical characteristics and the planet atmosphere and is considered to be a constant (Busemann and others, 1976:18).

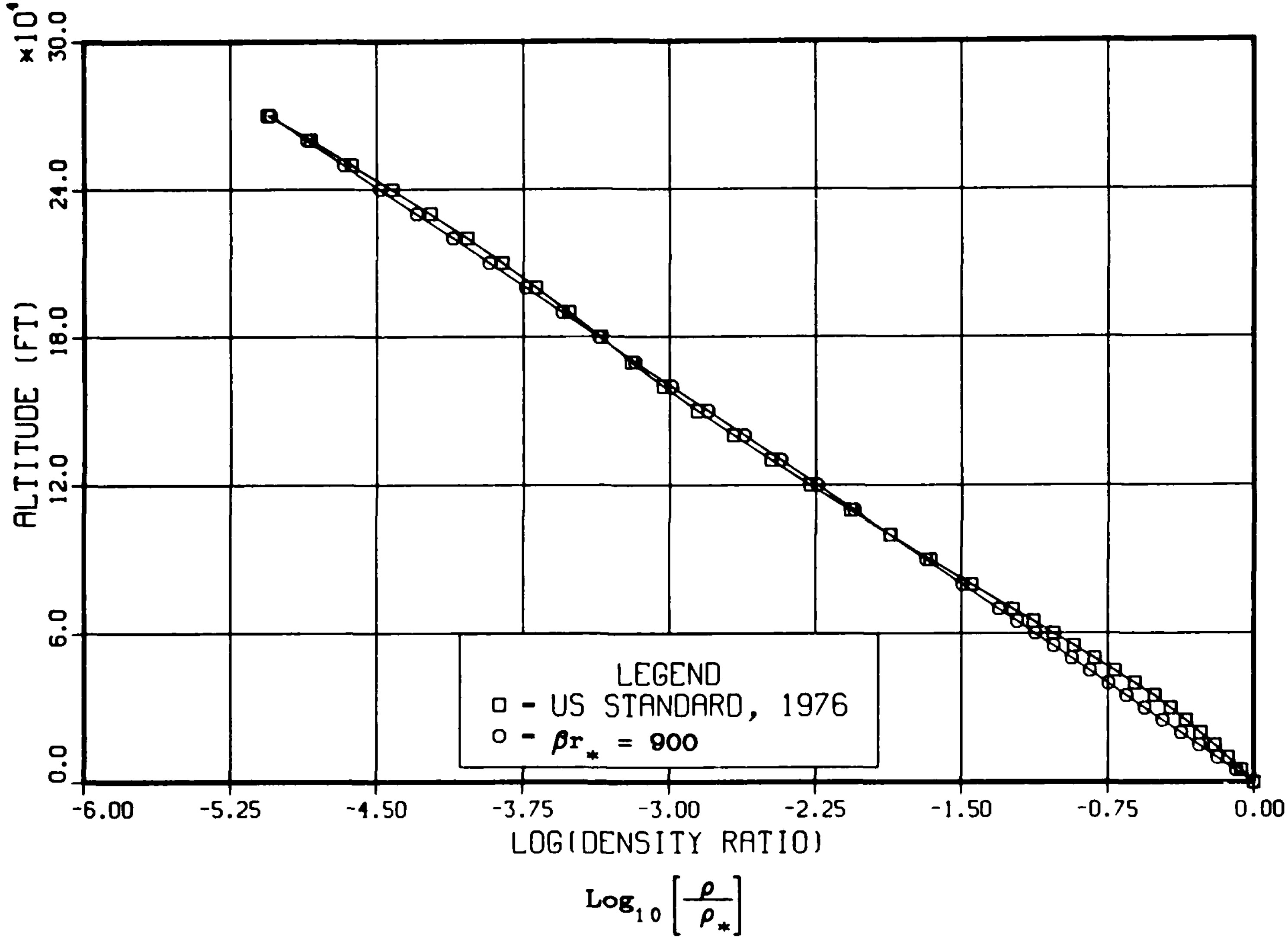

Figure 10. Earth Atmosphere Model Comparisons

The Speed Ratio. A non-dimensional variable is introduced to place the equations of motion into a more manageable form. This variable is the speed ratio, u, which is a Modified Chapman Variable used in many flight mechanics analyses. The dimensionless Chapman variable, $\bar{u}$, was utilized as the independent variable in the equations of motion in early analytical work on planetary entry. This variable was given by (Chapman, 1959:7)

$$\bar{u} = \frac{V\cos\gamma}{(gr)^{1/2}}$$

In later work (Buseman and others, 1976:11-13) it was found that $\bar{u}$ is periodic at high altitude, and that other dimensionless variables served better as the independent variable in the equations of motion. However, the Modified Chapman Variable, u, was found convenient to use in the derivation and analysis of the equations for planetary entry. For brevity, this Modified Chapman Variable is termed the "speed ratio", and is defined as the local horizontal component of the vehicle's velocity (in the xyz reference frame) divided by the square of the circular orbital velocity.

$$u = \frac{V^2\cos^2\gamma}{gr} = \frac{(1+h)V^2\cos^2\gamma}{g_*r_*} \tag{3.24}$$

This equation can be rewritten as:

$$V = \frac{1}{\cos\gamma}\left[\frac{ug_* r_*}{(1+h)}\right]^{1/2} \tag{3.25}$$

Differentiating Eq (3.18) with respect to h gives

$$\frac{du}{dh} = \frac{2V(1+h)\cos^2\gamma}{g_* r_*} \cdot \frac{dV}{dh}$$

$$- \frac{2V^2\,(1+h)\cos\gamma\sin\gamma}{g_* r_*} \cdot \frac{d\gamma}{dh} + \frac{V^2\cos^2\gamma}{g_* r_*} \tag{3.26}$$

Substituting Eqs (3.20), (3.23), and (3.25) into Eq (3.22) gives a non-dimensional equation for the aerodynamic drag acceleration:

$$\frac{D}{m} = V^2 \frac{\rho_*}{2m} S C_D e^{-h/\epsilon} = \beta B V^2 e^{-h/\epsilon}$$

or

$$\frac{D}{m} = \frac{ug_* B\exp(-h/\epsilon)}{\epsilon\,(1+h)\cos^2\gamma} \tag{3.27}$$

Likewise

$$\frac{L}{m} = \frac{D}{m} \cdot \frac{C_L}{C_D} = \frac{C_L}{C_D} \cdot \frac{ug_* B\exp(-h/\epsilon)}{\epsilon\,(1+h)\cos^2\gamma} \tag{3.28}$$

Changing the Independent Variable

The equations of motion for a rotating planet are now rewritten using the expressions derived above. To change the independent variable from time, t, to non-dimensional altitude, h, the equations of motion are multiplied by Eq (3.16). Direct substitution into the dV/dt equation gives

$$\frac{dV}{dh} = -\frac{ug_*r_*B\exp(-h/\epsilon)}{\epsilon(1+h)\cos\gamma\sin\gamma}\left[\frac{(1+h)}{ug_*r_*}\right]^{1/2} - \frac{g_*r_*\cos\gamma}{(1+h)^2}\left[\frac{(1+h)}{ug_*r_*}\right]^{1/2}$$

$$+ \omega^2 r_*{}^2\frac{(1+h)}{\sin\gamma}\left[\frac{(1+h)}{ug_*r_*}\right]^{1/2}\cdot\cos\phi\cos\gamma(\cos\phi\sin\gamma - \sin\phi\sin\psi\cos\gamma)$$

This equation can be reduced to

$$\frac{dV}{dh} = -\frac{B\exp(-h/\epsilon)}{\epsilon\cos\gamma\sin\gamma}\left[\frac{ug_*r_*}{(1+h)}\right]^{1/2} - \cos\gamma\left[\frac{g_*r_*}{u(1+h)^3}\right]^{1/2}$$

$$+ \omega^2\frac{\cos\phi}{\tan\gamma}\cdot\left[\frac{(1+h)^3 r_*{}^3}{ug_*}\right]^{1/2}\cdot(\cos\phi\sin\gamma - \sin\phi\sin\psi\cos\gamma) \qquad (3.29)$$

Direct substitution into the $d\gamma/dt$ equation gives

$$\frac{d\gamma}{dh} = \frac{\cos\gamma}{r_*(1+h)}\cdot\frac{r_*}{\sin\gamma} + \frac{C_L}{C_D}\cdot\frac{ug_*B\exp(-h/\epsilon)}{\epsilon(1+h)\cos^2\gamma}\cdot\frac{\cos\sigma}{\sin\gamma}\cdot\frac{r_*(1+h)\cos^2\gamma}{ug_*r_*}$$

$$-\frac{g_*\cos\gamma\ r_*(1+h)\cos^2\gamma}{(1+h)^2\ \sin\gamma\ ug_*r_*} + 2\omega\frac{\cos\phi\cos\psi}{\sin\gamma}r_*\cos\gamma\left[\frac{(1+h)}{ug_*r_*}\right]^{1/2}$$

$$+ \omega^2\frac{r_*^2(1+h)\cos\phi}{\sin\gamma}\cdot(\sin\phi\sin\psi\sin\gamma + \cos\gamma\cos\phi)\cdot\frac{(1+h)\cos^2\gamma}{ug_*r_*}$$

This equation can be reduced to

$$\frac{d\gamma}{dh} = \frac{1}{(1+h)\tan\gamma} + \frac{C_L}{C_D}\cdot\frac{B\cos\sigma}{\epsilon\sin\gamma}e^{-h/\epsilon} - \frac{\cos^2\gamma}{(1+h)u\tan\gamma}$$

$$+ 2\omega\left[\frac{(1+h)r_*}{ug_*}\right]^{1/2}\cdot\frac{\cos\phi\cos\psi}{\tan\gamma}$$

$$+ \omega^2\frac{r_*(1+h)^2}{ug_*}\cdot\frac{\cos\phi\cos\gamma}{\tan\gamma}(\sin\phi\sin\psi\sin\gamma + \cos\gamma\cos\phi) \qquad (3.30)$$

Direct substitution into the $d\psi/dt$ equation, Eq (3.3), gives

$$\frac{d\psi}{dh} = \frac{C_L}{C_D} \cdot \frac{ug_* B\exp(-h/\epsilon)}{\epsilon(1+h)\cos^2\gamma} \cdot \frac{\sin\sigma}{\cos\gamma} \cdot \frac{r_*\cos^2\gamma}{\sin\gamma} \left[\frac{(1+h)}{ur_* g_*}\right] - \frac{\cos\gamma\cos\psi\tan\phi}{r_*(1+h)\sin\gamma} r_*$$

$$+ \frac{2\omega r_*\cos\gamma}{\sin\gamma} \left[\frac{(1+h)}{ug_* r_*}\right]^{1/2} \cdot (\cos\phi\tan\gamma\sin\psi - \sin\phi)$$

$$- \frac{\omega^2 r_*(1+h)^2 \sin\phi\cos\phi\cos\psi}{\cos\gamma\sin\gamma} \cdot \frac{r_*(1+h)\cos^2\gamma}{ug_* r_*}$$

This reduces to

$$\frac{d\psi}{dh} = \frac{C_L}{C_D} \cdot \frac{Be^{-h/\epsilon}\sin\sigma}{\epsilon\cos\gamma\sin\gamma} - \frac{\cos\gamma\cos\psi\tan\phi}{(1+h)\sin\gamma} + \frac{2\omega}{\tan\gamma}\left[\frac{(1+h)r_*}{ug_*}\right]^{1/2} \cdot$$

$$\cdot (\cos\phi\tan\gamma\sin\psi - \sin\phi) - \frac{\omega^2 r_*(1+h)^2 \sin\phi\cos\phi\cos\psi}{ug_*\tan\gamma} \tag{3.31}$$

The $d\theta/dt$ equation becomes

$$\frac{d\theta}{dh} = \frac{\cos\psi}{(1+h)\cos\phi\tan\gamma} \tag{3.32}$$

and the $d\phi/dt$ equation becomes

$$\frac{d\phi}{dh} = \frac{\sin\psi}{(1+h)\tan\gamma} \tag{3.33}$$

The dr/dt equation is incorporated into the other equations, reducing the number of equations of motion from six to five. The du/dh equation also incorporates the dV/dh equation. Combining Eqns (3.19) and (3.20) allows the du/dh equation to be rewritten as

$$\frac{du}{dh} = 2\cos\gamma\left[\frac{u(1+h)}{r_* g_*}\right]^{1/2}\cdot\frac{dV}{dh} + \frac{u}{(1+h)} - 2u\tan\gamma\cdot\frac{d\gamma}{dh}$$

Substituting in for the dV/dh and dγ/dh terms using Eqs (3.29) and (3.30) eliminates V and gives the du/dh equation as a function of u.

$$\frac{du}{dh} = \frac{u}{(1+h)} + 2\cos\gamma\left[\frac{u(1+h)}{r_* g_*}\right]^{1/2}\cdot\left[\frac{-B\exp(-h/\epsilon)}{\epsilon\cos\gamma\sin\gamma}\left[\frac{ug_* r_*}{(1+h)}\right]^{1/2}\right.$$

$$- \cos\gamma\left[\frac{g_* r_*}{u(1+h)^3}\right]^{1/2} + \omega^2\frac{\cos\phi}{\tan\gamma}\cdot\left[\frac{(1+h)^3 r_*^{\,3}}{ug_*}\right]^{1/2}\cdot$$

$$\left.\cdot(\cos\phi\sin\gamma - \sin\phi\sin\psi\cos\gamma)\right]$$

$$- 2u\tan\gamma\cdot\left[\frac{1}{(1+h)\tan\gamma} + \frac{C_L}{C_D}\cdot\frac{B\cos\sigma}{\epsilon\sin\gamma}e^{-h/\epsilon} - \frac{\cos^2\gamma}{(1+h)u\tan\gamma}\right.$$

$$+ 2\omega\left[\frac{(1+h)r_*}{ug_*}\right]^{1/2}\frac{\cos\phi\cos\psi}{\tan\gamma}$$

$$\left.+ \omega^2\frac{r_*(1+h)^2}{ug_*}\cdot\frac{\cos\phi\cos\gamma}{\tan\gamma}(\sin\phi\sin\psi\sin\gamma + \cos\gamma\cos\phi)\right]$$

This rather formidable equation reduces to the following

$$\frac{du}{dh} = \frac{-u}{(1+h)} - \frac{2uBe^{-h/\epsilon}}{\epsilon}\left[\frac{1}{\sin\gamma} + \frac{C_L}{C_D}\cdot\frac{\cos\sigma}{\cos\gamma}\right]$$

$$- 4\omega\left[\frac{(1+h)ur_*}{g_*}\right]^{1/2}\cdot\cos\phi\cos\psi + 2\omega^2(1+h)^2\frac{r_*}{g_*}\cos\gamma\cos\phi\cdot$$

$$\cdot\left[\frac{\cos\phi\sin\gamma}{\tan\gamma} - \frac{\sin\phi\sin\psi\cos\gamma}{\tan\gamma} - \sin\phi\sin\psi\sin\gamma - \cos\phi\cos\gamma\right]$$

The du/dh equation can be further reduced to

$$\frac{du}{dh} = \frac{-u}{(1+h)} - \frac{2uBe^{-h/\epsilon}}{\epsilon\sin\gamma}\left[1 + \frac{C_L}{C_D}\cdot\tan\gamma\cos\sigma\right] - 4\omega\left[\frac{(1+h)ur_*}{g_*}\right]^{1/2}\cdot$$

$$\cdot\cos\phi\cos\psi - 2\omega^2\frac{(1+h)^2 r_*}{g_*}\cdot\frac{\cos\phi\sin\phi\sin\psi}{\tan\gamma} \tag{3.34}$$

Another variable is introduced to simplify the equations (Busemann and others, 1976:19).

$$\text{let} \quad q = \cos\gamma \quad \text{hence,} \quad \frac{dq}{dh} = -\sin\gamma\frac{d\gamma}{dh} \tag{3.35}$$

Using these relationships, the $d\gamma/dh$ equation can be rewritten

$$\frac{dq}{dh} = \frac{q}{(1+h)}\left[\frac{q^2}{u} - 1\right] - \frac{C_L}{C_D}\cdot\frac{Be^{-h/\epsilon}}{\epsilon}\cos\sigma - 2q\omega\cos\phi\cos\psi\left[\frac{r_*(1+h)}{ug_*}\right]^{1/2}$$

$$- \omega^2 r_* q^3 (1+h)^2\cdot\frac{\cos\phi}{ug_*}(\sin\phi\sin\psi\tan\gamma + \cos\phi) \tag{3.36}$$

The equations of motion for three-dimensional, rotating planetary entry have now been transformed from Eqs (3.1) - (3.6) to the following:

$$\frac{du}{dh} = \frac{-u}{(1+h)} - \frac{2uBe^{-h/\epsilon}}{\epsilon\sin\gamma}\left[1 + \frac{C_L}{C_D}\cdot\tan\gamma\cos\sigma\right] - 4\omega\left[\frac{(1+h)ur_*}{g_*}\right]^{1/2}\cdot$$

$$\cdot\cos\phi\cos\psi - 2\omega^2\frac{(1+h)^2 r_*}{g_*}\cdot\frac{\cos\phi\sin\phi\sin\psi}{\tan\gamma} \tag{3.37}$$

$$\frac{dq}{dh} = \frac{q}{(1+h)}\left[\frac{q^2}{u} - 1\right] - \frac{C_L}{C_D}\cdot\frac{Be^{-h/\epsilon}}{\epsilon}\cos\sigma - 2q\omega\cos\phi\cos\psi\left[\frac{r_*(1+h)}{ug_*}\right]^{1/2}$$

$$- \omega^2 r_* q^3 (1+h)^2 \cdot \frac{\cos\phi}{ug_*}(\sin\phi\sin\psi\tan\gamma + \cos\phi) \qquad (3.38)$$

$$\frac{d\psi}{dh} = \frac{C_L}{C_D} \cdot \frac{Be^{-h/\epsilon}\sin\sigma}{\epsilon\cos\gamma\sin\gamma} - \frac{\cos\gamma\cos\psi\tan\phi}{(1+h)\sin\gamma} + \frac{2\omega}{\tan\gamma}\left[\frac{(1+h)r_*}{ug_*}\right]^{1/2} \cdot$$

$$\cdot(\cos\phi\tan\gamma\sin\psi - \sin\phi) - \frac{\omega^2 r_* (1+h)^2 \sin\phi\cos\phi\cos\psi}{ug_* \tan\gamma} \qquad (3.39)$$

$$\frac{d\theta}{dh} = \frac{\cos\psi}{(1+h)\cos\phi\tan\gamma} \qquad (3.40)$$

$$\frac{d\phi}{dh} = \frac{\sin\psi}{(1+h)\tan\gamma} \qquad (3.41)$$

The Classical Orbit Variables

Up to this point, the equations of motion have been presented as functions of latitude, longitude, heading angle, and other variables. This form has been often utilized for atmospheric trajectory simulation by numerical integration. For ease in studying and in deriving solutions in later sections, these equations are placed in terms of the classical orbital elements, Ω, I, and α. For a non-rotating, spherical planet, these variables are constants of motion for non-atmospheric flight. This characteristic greatly simplifies the solution derivation for the non-rotating planet case. I is defined as the orbital inclination angle, Ω is the longitude of the ascending node, and α is the argument of latitude at epoch. Basic spherical trigonometric relations are found in Appendix A and these relations are applied to transform the variables θ, ϕ, and ψ

in terms of the orbital elements, α, Ω, and I. Figures 11 and 12 show the geometry of the two sets of variables.

The following relations, derived in Appendix A, relate θ, ϕ, and ψ, and α, Ω, and I.

$$\sin\phi = \sin I \sin\alpha \tag{3.42}$$

$$\cos I = \cos\phi\cos\psi \tag{3.43}$$

$$\sin\psi = \frac{\tan\phi}{\tan\alpha} \tag{3.44}$$

$$\sin(\theta-\Omega) = \frac{\tan\phi}{\tan I} \tag{3.45}$$

$$\sin\psi = \sin I\cos(\theta-\Omega) \tag{3.46}$$

$$\cos\alpha = \cos\phi\cos(\theta-\Omega) \tag{3.47}$$

Differentiating Eq (3.43) gives:

$$\sin I \cdot dI = \cos\phi\sin\psi \cdot d\psi + \sin\phi\cos\psi \cdot d\phi$$

Therefore

$$\frac{dI}{dh} = \frac{\cos\phi\sin\psi}{\sin I}\cdot\frac{d\psi}{dh} + \frac{\sin\phi\cos\psi}{\sin I}\cdot\frac{d\phi}{dh}$$

$$\frac{dI}{dh} = \frac{\cos\phi\sin\psi}{\sin I}\cdot\frac{d\psi}{dh} + \frac{\sin\phi\cos\psi\sin\psi}{\sin I\tan\gamma(1+h)} \tag{3.48}$$

Substituting for $d\psi/dh$, Eq (3.39), gives

$$\frac{dI}{dh} = \frac{C_L}{C_D}\cdot\frac{\cos\alpha\sin\sigma Be^{-h/\epsilon}}{\epsilon\cos\gamma\sin\gamma} - \frac{\cos\alpha\cos\gamma\cos\psi\tan\phi}{(1+h)\sin\gamma}$$

$$+ 2\omega\left[\frac{r_*(1+h)}{ug_*}\right]^{1/2}\frac{\cos\alpha}{\tan\gamma}(\cos\phi\tan\gamma\sin\psi - \sin\phi)$$

$$- \omega^2 r_*\cdot\frac{(1+h)^2}{ug_*\tan\gamma}\cos\alpha\sin\phi\cos\phi\cos\psi + \frac{\sin\phi\cos I\sin\psi}{\cos\phi(1+h)\tan\gamma\sin I} \tag{3.49}$$

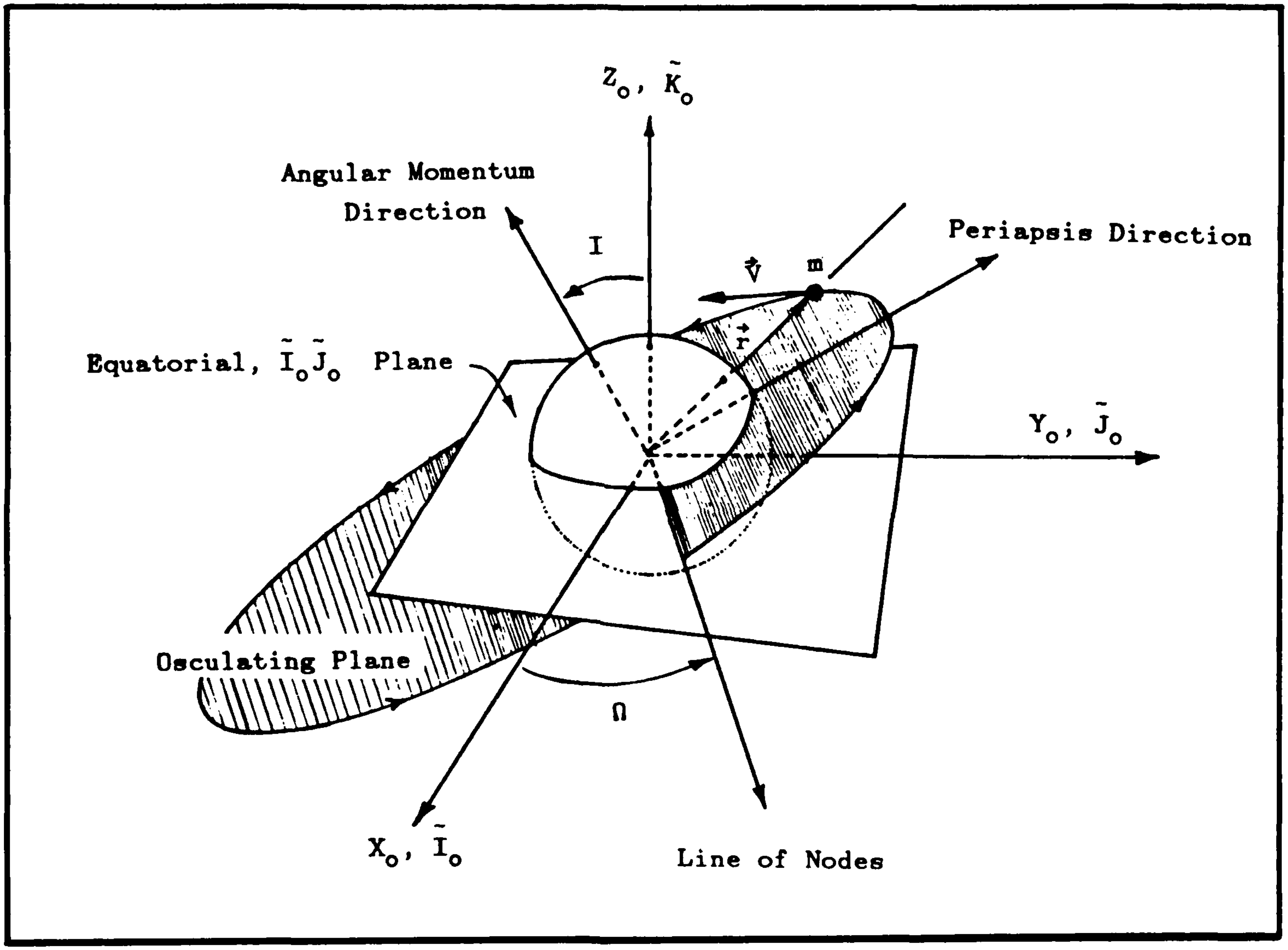

Figure 11. The Osculating and Equatorial Planes

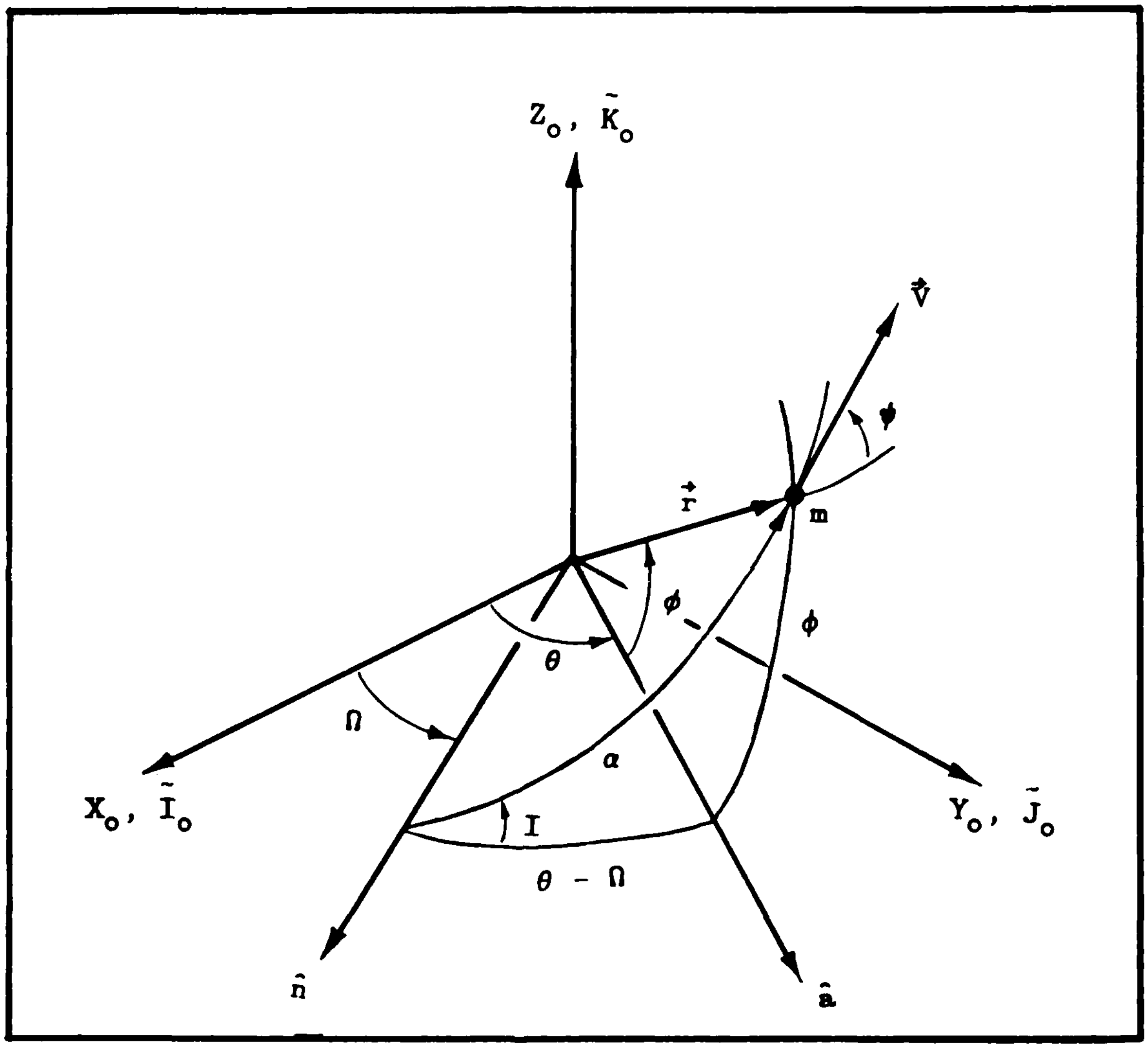

Figure 12. The Osculating Plane and the Orbital Elements

The last term cancels with the second term and with application of some of the spherical trigonometric equations, Eqs (3.42) - (3.47), the dI/dh equation is transformed from a function of θ, ϕ, and ψ, to a function of α, Ω, and I.

$$\frac{dI}{dh} = \frac{C_L}{C_D} \cdot \frac{\cos\alpha\sin\sigma B e^{-h/\epsilon}}{\epsilon\cos\gamma\sin\gamma} + 2\omega\left[\frac{r_*(1+h)}{ug_*}\right]^{1/2} \cdot \frac{\cos\alpha\sin I}{\tan\gamma} \cdot \tag{3.50}$$

$$\cdot(\cos\alpha\tan\gamma - \sin\alpha) - \omega^2 r_* \cdot \frac{(1+h)^2}{ug_*\tan\gamma}(\sin I\cos I\cos\alpha\sin\alpha)$$

To find an expression for $d\alpha/dh$ in terms of the desired variables, Eq (3.42) is differentiated.

$$\frac{d\phi}{dh} = \frac{\cos I\sin\alpha}{\cos\phi} \cdot \frac{dI}{dh} + \frac{\sin I\cos\alpha}{\cos\phi} \cdot \frac{d\alpha}{dh}$$

Substituting for $d\phi/dh$ with Eq (3.33) produces

$$\frac{d\alpha}{dh} = \frac{1}{(1+h)\tan\gamma} - \frac{\tan\alpha}{\tan I} \cdot \frac{dI}{dh} \tag{3.51}$$

$$\frac{d\alpha}{dh} = \frac{1}{(1+h)\tan\gamma} - \frac{C_L}{C_D} \cdot \frac{\sin\alpha\sin\sigma B e^{-h/\epsilon}}{\epsilon\cos\gamma\sin\gamma\tan I}$$

$$- 2\omega\left[\frac{r_*(1+h)}{ug_*}\right]^{1/2} \frac{\sin\alpha\cos I}{\tan\gamma}(\cos\alpha\tan\gamma - \sin\alpha)$$

$$+ \omega^2 r_* \cdot \frac{(1+h)^2}{ug_*\tan\gamma}\sin^2\alpha\cos^2 I \tag{3.52}$$

To find an expression for $d\Omega/dh$ in terms of the desired variables, Eq (3.46) is differentiated.

$$\sin\psi = \sin I\cos(\theta - \Omega)$$

$$\cos\psi\cdot d\psi = -\sin I\sin(\theta-\Omega)\cdot[d\theta-d\Omega] + \cos I\cos(\theta-\Omega)\cdot dI$$

By substituting in Eqs (3.32) and (3.45), this expression can be rewritten as

$$\frac{d\psi}{dh} = \frac{\cos I\tan\phi}{\cos\psi}\cdot\frac{d\Omega}{dh} - \frac{\cos I\tan\phi}{(1+h)\tan\gamma\cos\phi} + \frac{\cos I\cos\alpha}{\cos\psi\cos\phi}\cdot\frac{dI}{dh} \tag{3.53}$$

Rewriting Eq (3.48) gives another expression for $d\psi/dh$:

$$\frac{d\psi}{dh} = \frac{\sin I}{\cos\phi\sin\psi}\cdot\frac{dI}{dh} - \frac{\sin\phi\cos\psi}{(1+h)\tan\gamma} \tag{3.54}$$

Equating Eq (3.53) and Eq (3.54) gives $d\Omega/dh$ as a function of dI/dh.

$$\frac{d\Omega}{dh} = \frac{\cos\psi}{\cos I\tan\phi}\left[\frac{\cos I\tan\phi}{(1+h)\tan\gamma\cos\phi} - \frac{\sin\phi\cos\psi\cos\phi}{(1+h)\tan\gamma}\right.$$
$$\left. + \frac{dI}{dh}\cdot\left[\frac{\sin I}{\cos\phi\sin\psi} - \frac{\cos I\cos\alpha}{\cos\psi\cos\phi}\right]\right]$$

$$\frac{d\Omega}{dh} = \frac{\cos\psi}{\cos I\tan\phi\tan\phi(1+h)\tan\phi}\left[\frac{\cos I\tan\phi}{\cos\phi} - \frac{\cos I\sin\phi}{\cos^2\phi}\right]$$
$$+ \frac{dI}{dh}\cdot\frac{\cos\psi}{\cos I\sin\phi}\left[\frac{\cos\phi}{\cos\alpha} - \cos\phi\cos\alpha\right]$$

This simplifies to the following

$$\frac{d\Omega}{dh} = \frac{\tan\alpha}{\sin I}\cdot\frac{dI}{dh} \tag{3.55}$$

Eq (3.55) can be "de-simplified" by substituting Eq (3.50) into this equation.

$$\frac{d\Omega}{dh} = \frac{C_L}{C_D}\cdot\frac{Be^{-h/\epsilon}\sin\sigma\sin\alpha}{\epsilon\sin\gamma\cos\gamma\sin I} + \frac{2\omega}{\tan\gamma}\left[\frac{r_*(1+h)}{ug_*}\right]^{1/2}.$$

$$\cdot\left[\sin\alpha\cos\alpha\tan\gamma - \sin^2\alpha\right] - \frac{\omega^2 r_*(1+h)^2}{ug_*\tan\gamma}\cdot\cos I\sin^2\alpha \qquad (3.56)$$

The du/dh equation, Eq (3.34), can easily be transformed to the desired orbital variables by application of the spherical trigonometric relations.

$$\frac{du}{dh} = -\frac{u}{(1+h)} - \frac{2uB}{\epsilon\sin\gamma}e^{-h/\epsilon}\cdot\left[1 + \frac{C_L}{C_D}\cdot\cos\sigma\tan\gamma\right]$$

$$- 4\omega\cos I\left[\frac{ur_*(1+h)}{g_*}\right]^{1/2} - 2\omega^2 r_*\cdot\frac{(1+h)^2\cos\alpha\sin^2 I\sin\alpha}{g_*\tan\gamma} \qquad (3.57)$$

The last equation of motion to be transformed is the dq/dh equation, Eq (3.38). The ω^2 term of Eq (3.38) is

$$- \omega^2 r_* q^3\frac{(1+h)^2}{ug_*}\left[\cos\phi\sin\phi\sin\psi\tan\gamma + \cos^2\phi\right]$$

This term can be rewritten as

$$- \omega^2 r_* q^3\frac{(1+h)^2}{ug_*}\left[\cos\alpha\sin\alpha\sin^2 I\tan\gamma + 1 - \sin^2 I\sin^2\alpha\right]$$

The dq/dh equation transformed into the orbital elements is

$$\frac{dq}{dh} = \frac{q}{(1+h)}\left[\frac{q^2}{u} - 1\right] - \frac{C_L}{C_D}\cdot\frac{Be^{-h/\epsilon}}{\epsilon}\cos\sigma - 2q\omega\cos I\left[\frac{r_*(1+h)}{ug_*}\right]^{1/2} +$$

$$- \omega^2 r_* q^3\frac{(1+h)^2}{ug_*}\left[\sin\alpha\cos\alpha\tan\gamma\sin^2 I + 1 - \sin^2 I\sin^2\alpha\right] \qquad (3.58)$$

In summary, the equations of motion for three-dimensional rotating planetary entry have been derived for the independent variable, non-dimensional altitude, and some convenient dependent variables including orbital inclination, longitude of the ascending node, and argument of latitude at epoch. These equations are given below.

$$\frac{du}{dh} = -\frac{u}{(1+h)} - \frac{2Bu}{\epsilon\sin\gamma}\cdot e^{-h/\epsilon}\left[1 + \frac{C_L}{C_D}\cdot\cos\sigma\tan\gamma\right]$$

$$- 4\omega\cos I\left[\frac{ur_*(1+h)}{g_*}\right]^{1/2} - 2\omega^2 r_*\cdot\frac{(1+h)^2\cos\alpha\sin^2 I\sin\alpha}{g_*\tan\gamma} \qquad (3.59)$$

$$\frac{dq}{dh} = \frac{q}{(1+h)}\left[\frac{q^2}{u} - 1\right] - \frac{C_L}{C_D}\cdot\frac{Be^{-h/\epsilon}}{\epsilon}\cdot\cos\sigma - 2q\omega\cos I\left[\frac{r_*(1+h)}{ug_*}\right]^{1/2}$$

$$- \omega^2 r_* q^3\frac{(1+h)^2}{ug_*}\left[\sin\alpha\cos\alpha\tan\gamma\sin^2 I + 1 - \sin^2 I\sin^2\alpha\right] \qquad (3.60)$$

$$\frac{dI}{dh} = \frac{C_L}{C_D}\cdot\frac{\cos\alpha\sin\sigma Be^{-h/\epsilon}}{\epsilon\cos\gamma\sin\gamma} + 2\omega\left[\frac{r_*(1+h)}{ug_*}\right]^{1/2}\cdot$$

$$\cdot\frac{\cos\alpha\sin I}{\tan\gamma}(\cos\alpha\tan\gamma - \sin\alpha)$$

$$- \omega^2 r_*\cdot\frac{(1+h)^2}{ug_*\tan\gamma}(\sin I\cos I\cos\alpha\sin\alpha) \qquad (3.61)$$

$$\frac{d\Omega}{dh} = \frac{C_L}{C_D}\cdot\frac{Be^{-h/\epsilon}\sin\sigma\sin\alpha}{\epsilon\sin\gamma\cos\gamma\sin I} + \frac{2\omega}{\tan\gamma}\left[\frac{r_*(1+h)}{ug_*}\right]^{1/2}\cdot$$

$$\cdot\left[\sin\alpha\cos\alpha\tan\gamma - \sin^2\alpha\right] - \frac{\omega^2 r_*(1+h)^2}{ug_*\tan\gamma}\cdot\cos I\sin^2\alpha \qquad (3.62)$$

$$\frac{d\alpha}{dh} = \frac{1}{(1+h)\tan\gamma} - \frac{C_L}{C_D}\cdot\frac{\sin\alpha\sin\sigma Be^{-h/\epsilon}}{\epsilon\cos\gamma\sin\gamma\tan I}$$

$$- 2\omega\left[\frac{r_*(1+h)}{ug_*}\right]^{1/2}\frac{\sin\alpha\cos I}{\tan\gamma}(\cos\alpha\tan\gamma - \sin\alpha)$$

$$+ \omega^2 r_* \cdot \frac{(1+h)^2}{u g_* \tan\gamma} \sin^2\alpha \cos^2 I \tag{3.63}$$

Singularities exist for these equations of motion for flight path angle values of 0.0 and 90.0 degrees and for an orbital inclination angle of 0.0 degrees. Special consideration must be taken when evaluating Eqs (3.59) - (3.63) near these values.

IV. Examination of the Rotating Planet Terms

The rotating Earth terms in each of the equations of motion are examined in this section. This examination is conducted for orbital inclination angles ranging from 0.5 to 75.0 degrees and vehicle speeds ranging from circular orbital velocity to low supersonic speeds, where terminal maneuvers such as landing approaches are usually initiated. The rotating Earth terms are set equal to zero for each equation of motion and are then checked for the existance of real solutions. The existance of real solutions for any of these expressions indicates trajectory states (specific combinations of values of u, h, q, I, Ω, and α along an entry trajectory) exist where that particular non-rotating Earth equation of motion is valid. In a later section, these trajectory states will be examined in more detail. The non-existance of real solutions to the rotating Earth terms in any one of the equations of motion indicates the corresponding non-rotating equation of motion will be invalid for any Earth entry trajectory. A solution that accounts for the Earth's rotation is needed for any non-rotating Earth equation of motion that is invalid for the full range of inclination angle and speeds investigated. These solutions are developed in Section V by application of the method of matched asymptotic expansions.

The Second Small Parameter

To begin, the equations of motion are placed in a more convenient form to work with. In Section III, ϵ was shown to be approximately equal to the ratio of the atmospheric scale height and the radius of the planet. A convenient second small parameter is introduced to replace the ω terms in the equations of motion. This second small parameter is defined as the square of the ratio of the planet's rotational velocity at the equator and the prograde, equatorial circular orbital velocity at the surface. The orginal small parameter, ϵ, is re-labelled ϵ_1 and the second small parameter is labelled ϵ_2 .

$$\epsilon_1 = \frac{1}{\beta r_*} \quad \text{and} \quad \epsilon_2 = \left[\frac{\omega r_*}{(g_* r_*)^{1/2}}\right]^2 = \omega^2 \frac{r_*}{g_*} \tag{4.1}$$

For Earth, $\epsilon_1 = 1/900$ and $\epsilon_2 = 1/289$

The equations of motion for three-dimensional, rotating planetary entry, derived in Section II, now become

$$\frac{du}{dh} = \frac{-u}{(1+h)} - \frac{2uBe^{-h/\epsilon_1}}{\epsilon_1 \sin\gamma}\cdot\left[1 + \frac{C_L}{C_D}\cdot\tan\gamma\cos\sigma\right]$$

$$- 4\left[\epsilon_2 u(1+h)\right]^{1/2}\cdot\cos I - 2\epsilon_2 (1+h)^2 \frac{\cos\alpha\sin^2 I\sin\alpha}{\tan\gamma} \tag{4.2}$$

$$\frac{dq}{dh} = \frac{q}{(1+h)}\left[\frac{q^2}{u} - 1\right] - \frac{C_L}{C_D}\cdot\frac{Be^{-h/\epsilon_1}}{\epsilon_1}\cos\sigma - 2q\left[\epsilon_2\frac{(1+h)}{u}\right]^{1/2}\cos I$$

$$- \epsilon_2 (1+h)^2 \cdot \frac{q^3}{u} (\tan\gamma \sin\alpha \cos\alpha \sin^2 I + 1 - \sin^2 I \sin^2 \alpha) \quad (4.3)$$

$$\frac{d\alpha}{dh} = \frac{1}{(1+h)\tan\gamma} - \frac{C_L}{C_D} \cdot \frac{Be^{-h/\epsilon_1}}{\epsilon_1} \cdot \frac{\sin\sigma \sin\alpha}{\cos\gamma \sin\gamma \tan I}$$

$$- 2\left[\epsilon_2 \frac{(1+h)}{u}\right]^{1/2} \cdot \frac{\sin\alpha \cos I}{\tan\gamma} (\cos\alpha \tan\gamma - \sin\alpha)$$

$$+ \epsilon_2 (1+h)^2 \cdot \frac{\sin^2\alpha \cos^2 I}{u \tan\gamma} \quad (4.4)$$

$$\frac{d\Omega}{dh} = \frac{C_L}{C_D} \cdot \frac{Be^{-h/\epsilon_1}}{\epsilon_1} \frac{\sin\sigma \sin\alpha}{\cos\gamma \sin\gamma \sin I} + 2\left[\epsilon_2 \frac{(1+h)}{u}\right]^{1/2} \cdot \frac{\sin\alpha}{\tan\gamma} (\cos\alpha \tan\gamma - \sin\alpha)$$

$$- \epsilon_2 (1+h)^2 \cdot \frac{\sin^2\alpha \cos I}{u \tan\gamma} \quad (4.5)$$

$$\frac{dI}{dh} = \frac{C_L}{C_D} \cdot \frac{Be^{-h/\epsilon_1}}{\epsilon_1} \cdot \frac{\sin\sigma \cos\alpha}{\cos\gamma \sin\gamma} + 2\left[\epsilon_2 \frac{(1+h)}{u}\right]^{1/2} \cdot \frac{\cos\alpha \sin I}{\tan\gamma} \cdot$$

$$\cdot (\cos\alpha \tan\gamma - \sin\alpha) - \epsilon_2 (1+h)^2 \cdot \frac{\cos\alpha \sin\alpha \cos I \sin I}{u \tan\gamma} \quad (4.6)$$

It was previously noted that the latter three of the above equations are coupled:

$$\frac{d\Omega}{dh} = \frac{\tan\alpha}{\sin I} \cdot \frac{dI}{dh} \quad (4.7)$$

$$\frac{d\alpha}{dh} = \frac{1}{(1+h)\tan\gamma} - \frac{\tan\alpha}{\tan I} \cdot \frac{dI}{dh} \quad (4.8)$$

The Rotating Earth Terms

The equations of motion for lifting atmospheric entry for a rotating Earth can be written as the sum of the equations for a non-rotating Earth and of the terms that account for the Earth's rotation. The equations for a non-rotating Earth are the terms in Eqs (4.2) - (4.6) which do not contain ϵ_2, and hence are not a function of ω. The non-rotating Earth terms are referred to as the "Nonrotate equations" and the rotating Earth terms in the equations of motion are simply referred to as the "Rotate equation".

Speed Ratio Equation. The rotating Earth equation of motion for speed ratio, given by Eq (4.2), can be rewritten as

$$\frac{du}{dh} = \text{Nonrotate} + \text{Rotate}$$

where

$$\left.\frac{du}{dh}\right|_{\text{Nonrotate}} = \frac{-u}{(1+h)} - \frac{2uBe^{-h/\epsilon_1}}{\epsilon_1 \sin\gamma} \cdot \left[1 + \frac{C_L}{C_D} \cdot \tan\gamma\cos\sigma\right]$$

$$\left.\frac{du}{dh}\right|_{\text{Rotate}} = -4\left[\epsilon_2 u(1+h)\right]^{1/2} \cos I$$

$$- 2\epsilon_2 (1+h)^2 \frac{\cos\alpha\sin^2 I\sin\alpha}{\tan\gamma} \tag{4.9}$$

Flight Path Angle Equation. The rotating Earth equation of motion for flight path angle, given by Eq (4.3), can be rewritten as

$$\frac{dq}{dh} = \text{Nonrotate} + \text{Rotate}$$

where

$$\left.\frac{dq}{dh}\right|_{Nonrotate} = \frac{q}{(1+h)}\left[\frac{q^2}{u} - 1\right] - \frac{C_L}{C_D}\cdot\frac{Be^{-h/\epsilon_1}}{\epsilon_1}\cdot\cos\sigma$$

$$\left.\frac{dq}{dh}\right|_{Rotate} = -\ 2q\left[\epsilon_2\frac{(1+h)}{u}\right]^{1/2}\cos I \tag{4.10}$$

$$-\ \epsilon_2(1+h)^2\cdot\frac{q^3}{u}(\tan\gamma\sin\alpha\cos\alpha\sin^2 I + 1 - \sin^2 I\sin^2\alpha)$$

<u>Argument of Latitude at Epoch</u>. The rotating Earth equation of motion for argument of latitude at epoch, given by Eq (4.4), can be rewritten as

$$\frac{d\alpha}{dh} = \text{Nonrotate} + \text{Rotate}$$

where

$$\left.\frac{d\alpha}{dh}\right|_{Nonrotate} = \frac{1}{(1+h)\tan\gamma} - \frac{C_L}{C_D}\cdot\frac{Be^{-h/\epsilon_1}}{\epsilon_1}\cdot\frac{\sin\sigma\sin\alpha}{\cos\gamma\sin\gamma\tan I}$$

$$\left.\frac{d\alpha}{dh}\right|_{Rotate} = -\ 2\left[\epsilon_2\frac{(1+h)}{u}\right]^{1/2}\cdot\frac{\sin\alpha\cos I}{\tan\gamma}(\cos\alpha\tan\gamma - \sin\alpha)$$

$$+\ \epsilon_2(1+h)^2\cdot\frac{\sin^2\alpha\cos^2 I}{u\tan\gamma} \tag{4.11}$$

<u>Longitude of the Ascending Node</u>. The rotating Earth equation of motion for longitude of the ascending node, given by Eq (4.5), can be rewritten as

$$\frac{d\Omega}{dh} = \text{Nonrotate} + \text{Rotate}$$

where

$$\left.\frac{d\Omega}{dh}\right|_{Nonrotate} = \frac{C_L}{C_D}\cdot\frac{Be^{-h/\epsilon_1}}{\epsilon_1}\cdot\frac{\sin\sigma\sin\alpha}{\cos\gamma\sin\gamma\sin I}$$

$$\left.\frac{d\Omega}{dh}\right|_{Rotate} = 2\left[\epsilon_2\frac{(1+h)}{u}\right]^{1/2}\cdot\frac{\sin\alpha}{\tan\gamma}(\cos\alpha\tan\gamma - \sin\alpha)$$

$$- \epsilon_2(1+h)^2\cdot\frac{\sin^2\alpha\cos I}{u\tan\gamma} \qquad (4.12)$$

Inclination Angle Equation. The rotating Earth equation of motion for orbital inclination angle, given by Eq (4.6), can be rewritten as

$$\frac{dI}{dh} = \text{Nonrotate} + \text{Rotate}$$

where

$$\left.\frac{dI}{dh}\right|_{Nonrotate} = \frac{C_L}{C_D}\cdot\frac{Be^{-h/\epsilon_1}}{\epsilon_1}\cdot\frac{\sin\sigma\cos\alpha}{\cos\gamma\sin\gamma}$$

$$\left.\frac{dI}{dh}\right|_{Rotate} = 2\left[\epsilon_2\frac{(1+h)}{u}\right]^{1/2}\cdot\frac{\cos\alpha\sin I}{\tan\gamma}(\cos\alpha\tan\gamma - \sin\alpha)$$

$$- \epsilon_2(1+h)^2\cdot\frac{\cos\alpha\sin\alpha\cos I\sin I}{u\tan\gamma} \qquad (4.13)$$

Investigation of Real Solutions to the Rotate Terms

Eqs (4.9) - (4.13) are the Rotate equations, those parts of the complete equations of motion which account for the rotating Earth. In the following pages, the five Rotate equations are each set equal to zero and then examined for the existence of real solutions.

Rotate Term Solutions for the Speed Ratio Equation. From Eq (4.9)

$$\left.\frac{du}{dh}\right|_{Rotate} = -4\left[\epsilon_2 u(1+h)\right]^{1/2}\cos I - 2\epsilon_2 (1+h)^2 \frac{\cos\alpha \sin^2 I \sin\alpha}{\tan\gamma}$$

Setting this equation equal to zero and solving for one of the five variables gives

$$\left.\gamma\right|_{Rotate\ =\ 0} = \tan^{-1}\left[\frac{-C\cos\alpha\sin\alpha\sin^2 I}{2\cos I}\right] \tag{4.14}$$

where $$C = \left[\frac{(1+h)^3}{u}\epsilon_2\right]^{1/2} \tag{4.15}$$

Substitution of a few realistic combinations of u, h, I, and α indicate real solutions exist for Eq (4.14).

Rotate Term Solutions for the Inclination Angle Equation. From Eq (4.13)

$$\left.\frac{dI}{dh}\right|_{Rotate} = 2\left[\epsilon_2 \frac{(1+h)}{u}\right]^{1/2} \cdot \frac{\cos\alpha \sin I}{\tan\gamma}(\cos\alpha\tan\gamma - \sin\alpha)$$

$$- \epsilon_2 (1+h)^2 \cdot \frac{\cos\alpha\sin\alpha\cos I\sin I}{u\tan\gamma}$$

Setting this equation equal to zero and substituting in Eq (4.15) for C gives

$$2\sin I\cos\alpha\left[\cos\alpha - \frac{\sin\alpha}{\tan\gamma}\right] = \frac{C}{\tan\gamma}\sin\alpha\cos\alpha\sin I\cos I \tag{4.16}$$

One set of trivial solutions to this equation is in the form

$\sin I\cos\alpha = 0$ for any C and γ

The ω term contribution to the dI/dh equation of motion is therefore equal to zero when sinI and/or $\cos\alpha = 0$. However, due to the fact the equations of motion are singular for $\sin I = 0$ as discussed in Section III, the trivial solution is actually

$\cos\alpha = 0 \quad (\alpha = \ldots -3\pi/2,\ -\pi/2,\ \pi/2,\ 3\pi/2,\ \ldots)$

Nontrivial solutions to Eq (4.16) can be found by solving for one of the five variables.

$$\gamma\Big|_{\text{Rotate} = 0} = \tan^{-1}\left[\frac{\sin\alpha + C\cos I\sin\alpha}{2\cos\alpha}\right] \tag{4.17}$$

Substitution of a few realistic combinations of u, h, I, and α indicate real solutions exist for Eq (4.17).

<u>Rotate Term Solutions for the Argument of Latitude at Epoch</u>. From Eq (4.11)

$$\frac{d\alpha}{dh}\Big|_{\text{Rotate}} = -2\left[\epsilon_2\frac{(1+h)}{u}\right]^{1/2}\cdot\frac{\sin\alpha\ \cos I}{\tan\gamma}(\cos\alpha\tan\gamma - \sin\alpha)$$

$$+\ \epsilon_2(1+h)^2\cdot\frac{\sin^2\alpha\ \cos^2 I}{u\tan\gamma}$$

Setting this equation equal to zero and substituting in Eq (4.15) for C gives

$$2\sin\alpha\cos I\left[\cos\alpha - \frac{\sin\alpha}{\tan\gamma}\right] = \frac{C}{\tan\gamma}\cdot\sin^2\alpha\cos^2 I \tag{4.18}$$

One set of trivial solutions to this equation is in the form

$\sin\alpha\cos I = 0 \quad$ for any C and γ

The ω term contribution to the $d\alpha/dh$ equation of motion is hence equal to zero when $\sin\alpha$ and/or $\cos I$ are zero. However, due to the range of inclination angle selected for study, $\cos I$ is never equal to zero and the trivial solution is

$\sin\alpha = 0 \quad (\alpha = \ldots -2\pi, -\pi, 0, \pi, 2\pi, \ldots)$

Nontrivial solutions to Eq (4.18) can be found by solving for one of the five variables in this equation.

$$\gamma\Big|_{\text{Rotate} = 0} = \tan^{-1}\left[\frac{C\cos I\sin\alpha + \sin\alpha}{2\cos\alpha}\right] \tag{4.19}$$

Substitution of a few realistic combinations of u, h, I, and α indicate real solutions exist for Eq (4.19).

<u>Rotate Term Solutions for the Longitude of the Ascending Node</u>. From Eq (4.12)

$$\frac{d\Omega}{dh}\Big|_{\text{Rotate}} = 2\left[\epsilon_2\frac{(1+h)}{u}\right]^{1/2}\cdot\frac{\sin\alpha}{\tan\gamma}(\cos\alpha\tan\gamma - \sin\alpha)$$

$$- \epsilon_2(1+h)^2\cdot\frac{\sin^2\alpha\cos I}{u\tan\gamma}$$

Setting this equation equal to zero and substituting in Eq (4.15) for C gives

$$2\sin\alpha\left[\cos\alpha - \frac{\sin\alpha}{\tan\gamma}\right] = C\sin^2\alpha\frac{\cos I}{\tan\gamma} \tag{4.20}$$

One set of trivial solutions to this equation is in the form

$\sin\alpha = 0$ for any C, I, and γ

Hence, the ω term contribution to the $d\alpha/dh$ equation of motion is equal to zero for $\alpha = \ldots -2\pi, -\pi, 0, \pi, 2\pi, \ldots$

Nontrivial solutions to Eq (4.20) are in the form

$$\gamma\Big|_{\text{Rotate} = 0} = \tan^{-1}\left[\frac{C\cos I\sin\alpha + \sin\alpha}{2\cos\alpha}\right] \tag{4.21}$$

Substitution of a few realistic combinations of u, h, I, and α indicate real solutions exist for Eq (4.21).

Coupling of Three Rotate Term Solutions. The nontrivial solutions for the rotating term expressions are identical for the $d\alpha/dh$, $d\Omega/dh$, and dI/dh equations, Eqs (4.17), (4.19), and (4.21). This can also be seen by noting that the dI/dh, $d\Omega/dh$, and $d\alpha/dh$ equations are coupled by Eqs (4.7) and (4.8).

$$\frac{d\Omega}{dh} = \frac{\tan\alpha}{\sin I}\cdot\frac{dI}{dh} \tag{4.7}$$

$$\frac{d\alpha}{dh} = \frac{1}{(1+h)\tan\gamma} - \frac{\tan\alpha}{\tan I}\cdot\frac{dI}{dh} \tag{4.8}$$

These coupling relations can be written in terms of the Nonrotate and Rotate expressions:

$$\frac{d\Omega}{dh}\Big|_{\substack{\text{Nonrotate}\\ +\text{ Rotate}}} = \frac{\tan\alpha}{\sin I}\cdot\frac{dI}{dh}\Big|_{\text{Nonrotate}} + \frac{\tan\alpha}{\sin I}\cdot\frac{dI}{dh}\Big|_{\text{Rotate}} \tag{4.22}$$

$$\frac{d\alpha}{dh}\Big|_{\substack{\text{Nonrotate}\\ +\text{ Rotate}}} = \frac{1}{(1+h)\tan\gamma} - \frac{\tan\alpha}{\tan I}\cdot\frac{dI}{dh}\Big|_{\text{Nonrotate}}$$

$$- \frac{\tan\alpha}{\tan I}\cdot\frac{dI}{dh}\Big|_{\text{Rotate}} \tag{4.23}$$

Since the Rotate terms contain ω or ϵ_2 by definition, the first term of Eq (4.23) must be a Nonrotate expression. Hence,

$$\left.\frac{d\alpha}{dh}\right|_{Rotate} = - \left.\frac{\tan\alpha}{\tan I}\cdot\frac{dI}{dh}\right|_{Rotate} \tag{4.24}$$

In addition

$$\left.\frac{d\Omega}{dh}\right|_{Rotate} = \left.\frac{\tan\alpha}{\sin I}\cdot\frac{dI}{dh}\right|_{Rotate} \tag{4.25}$$

From Eqs (4.7) and (4.8) it can be seen that the $d\Omega/dh$ and $d\alpha/dh$ equations differ significantly. However, comparing like terms indicates

$$\left.\frac{d\alpha}{dh}\right|_{Rotate} = - \left.\cos I\frac{d\Omega}{dh}\right|_{Rotate} \tag{4.26}$$

These results simplify the search for the trajectory states where the non-rotating Earth equations of motion are valid. Instead of detailed examination of the non-trivial solutions of all three of the $d\alpha/dh$, $d\Omega/dh$, and dI/dh Rotate equations, examination of only one of them is required. The trivial solutions for rotating term expressions for the $d\Omega/dh$ and $d\alpha/dh$ Rotate equations are the same, $\sin\alpha = 0$, and the trivial solution for the rotating term expression for the dI/dh Rotate equation is $\cos\alpha = 0$.

<u>Rotate Term Solutions for the Flight Path Angle Equation</u>. From Eq (4.10)

$$\left.\frac{dq}{dh}\right|_{Rotate} = -\ 2q\left[\epsilon_2 \frac{(1+h)}{u}\right]^{1/2} \cos I$$

$$-\ \epsilon_2 (1+h)^2 \cdot \frac{q^3}{u} (\tan\gamma \sin\alpha \cos\alpha \sin^2 I + 1 - \sin^2 I \sin^2 \alpha)$$

Setting this equation equal to zero gives

$$0 = 2qA_1 \cos I + q^3 \left[\tan\gamma \sin\alpha \cos\alpha \sin^2 I + 1 - \sin^2 I \sin^2 \alpha\right] \quad (4.27)$$

$$\text{where} \qquad A_1 = \left[\frac{u}{\epsilon_2 (1+h)^3}\right]^{1/2} \quad (4.28)$$

Because singularities exist for I or γ = 0.0 or 90.0 degrees, $q = 0$ can not be considered as a trivial solution. To examine possible solutions to Eq (4.27), some additional variables, A_i, are defined for convenience.

$$\text{let} \qquad A_2 = 1 - \sin^2 I \sin^2 \alpha \quad (4.29)$$

$$A_3 = \sin\alpha \cos\alpha \sin^2 I \quad (4.30)$$

$$A_4 = 2A_1 \cos I \quad (4.31)$$

Using these variables, dividing by q, and remembering $q = \cos\gamma$, equation Eq (4.27) can be rewritten

$$A_4 + q^2 A_3 \tan\gamma + q^2 A_2 = 0$$

$$A_4 + A_2 \cos^2\gamma = -\ A_3 \sin\gamma \cos\gamma \quad (4.32)$$

Squaring this expression

$$A_2{}^2\cos^4\gamma + A_4{}^2 + 2A_4A_2\cos^2\gamma - A_3{}^2\cos^2\gamma + A_3{}^2\cos^4\gamma = 0$$

$$(A_2{}^2 + A_3{}^2)\nu^2 + (2A_4A_2 - A_3{}^2)\nu + A_4{}^2 = 0 \tag{4.33}$$

$$\text{where} \quad \nu = q^2 = \cos^2\gamma \tag{4.34}$$

Eq (4.33) can be easily solved in the form

$$A\nu^2 + B\nu + C = 0$$

where $A = A_2{}^2 + A_3{}^2$, $B = 2A_4A_2 - A_3{}^2$, $C = A_4{}^2$, and

$$\nu = \frac{-B \pm \left[B^2 - 4AC\right]^{1/2}}{2A}$$

The Rotate term solution for the flight path angle equation is hence given by

$$\nu = \frac{-2A_4A_2 + A_3{}^2 \pm \left[A_3{}^4 - 4A_4A_2A_3^2 - 4A_3{}^2A_4{}^2\right]^{1/2}}{2A_2{}^2 + 2A_3{}^2} \tag{4.35}$$

At this point ν has been placed in terms of the variables A_1, A_2, A_3, and A_4. Looking at each of these expressions individually aids in the determination of general trends of the values of ν.

<u>A_2 Equation</u>. It can be easily seen that A_2 has a maximum value of 1.0 when $\sin\alpha = 0$, for any value of $\sin I$ within the inclination range of interest, $0.5 \le I \le 75.0$ degrees. A minimum of 0.0670 occurs for A_2 when I = 75.0

degrees and α = 90.0 degrees or 90.0 ± 180.0 degrees. Thus, A_2 is never negative.

$\underline{A_3 \text{ Equation}}$. A_3 has a maximum value of 0.466 for I = 75.0 degrees and α = 45.0 degrees. A minimum value of -0.466 occurs when I = 75.0 degrees and α = -45.0 degrees.

$\underline{A_4 \text{ Equation}}$. Since A_1 is always positive, it can be easily seen A_4 has a maximum value of 34.0 for I = 0.5 degrees, u = 1.0, and h = 0.0 . A_4 has a minimum of 0.0 for u = 0.0 . Minimum values of A_4 for non-zero u occur for I = 75 degrees. For a prograde orbit A_4 is always positive.

Solutions. For real solutions to exist in Eq (4.35), and hence for the rotating terms in the dq/dh equation to ever have a zero contribution, the expression within the square root must be positive. Factoring A_3 out of the root expression in Eq (4.35) leaves the following condition for the existence of real roots

$$A_3{}^2 - 4\left[A_2 A_4 + A_4{}^2\right] > 0 \qquad (4.36)$$

where A_2 and A_4 are always positive and A_3 can be positive or negative over the range of inclination angle examined.

Eq (4.36) indicates real solutions to the dq/dh Rotate equation do not exist for realistic entry trajectories except for values of u corresponding to low speed, subsonic and near subsonic flight. Therefore, no real roots to Eq (4.35) exist for orbital inclination angles between 0.5 and 75.0 degrees and vehicle speeds ranging from circular orbital velocities to low supersonic speeds. This result indicates the non-rotating Earth dq/dh equation of motion is invalid for Earth atmospheric entry for the ranges of orbital inclination angle and velocity investigated. Therefore, a first order solution (Section V) to the complete dq/dh equation is required for Earth atmospheric entry analysis. This solution is developed in the next section using the method of matched asymptotic expansions.

V. Solutions to the Equations of Motion Using Matched Asymptotic Expansions

In Section IV, the terms in the five equations of motion that account for Earth's rotation were examined. It was determined that trajectory states exist for four of the five equations of motion where the rotating Earth terms give a zero contribution. For the fifth equation of motion, the dq/dh equation, it was shown that the rotating earth terms always have a non-zero contribution for realistic lifting entry trajectories with entry inclination angles between 0.5 and 75.0 degrees and speeds ranging from circular orbital velocities to low supersonic speeds. Therefore, to adequately model Earth atmospheric entry, a solution to the dq/dh equation is required that includes the rotating Earth effects. This solution, along with the non-rotating Earth solutions to the other four equations of motion, is developed in this section, by application of the method of matched asymptotic expansions.

Combining the Small Parameters

In Section III, ϵ was defined to be the ratio of the planetary atmosphere scale height and the planet's radius. A second small parameter was introduced in Section IV to non-dimensionalize the rotating planet terms in the equations of motion. This second small parameter was defined as the square of the quotient of the planet's rotational velocity at the equator and the prograde, equatorial circular orbital velocity at the mean planetary radius.

$$\epsilon_1 = \frac{1}{\beta r_*} \quad \text{and} \quad \epsilon_2 = \left[\frac{\omega r_*}{(g_* r_*)^{1/2}}\right]^2 = \omega^2 \frac{r_*}{g_*} \tag{5.1}$$

The general equations of motion for three-dimensional, rotating planetary entry were given in Section IV in the form

$$\frac{du}{dh} = \frac{-u}{(1+h)} - \frac{2uBe^{-h/\epsilon_1}}{\epsilon_1 \sin\gamma}\cdot\left[1 + \frac{C_L}{C_D}\cdot\tan\gamma\cos\sigma\right]$$

$$- 4\left[\epsilon_2 u(1+h)\right]^{1/2}\cos I - 2\epsilon_2 (1+h)^2 \frac{\cos\alpha\sin^2 I\sin\alpha}{\tan\gamma} \tag{5.2}$$

$$\frac{dq}{dh} = \frac{q}{(1+h)}\left[\frac{q^2}{u} - 1\right] - \frac{C_L}{C_D}\cdot\frac{Be^{-h/\epsilon_1}}{\epsilon_1}\cos\sigma - 2q\left[\epsilon_2\frac{(1+h)}{u}\right]^{1/2}\cos I$$

$$- \epsilon_2 (1+h)^2\cdot\frac{q^3}{u}(\tan\gamma\sin\alpha\cos\alpha\sin^2 I + 1 - \sin^2 I\sin^2\alpha) \tag{5.3}$$

$$\frac{d\alpha}{dh} = \frac{1}{(1+h)\tan\gamma} - \frac{C_L}{C_D}\cdot\frac{Be^{-h/\epsilon_1}}{\epsilon_1}\cdot\frac{\sin\sigma\sin\alpha}{\cos\gamma\sin\gamma\tan I}$$

$$- 2\left[\epsilon_2\frac{(1+h)}{u}\right]^{1/2}\frac{\sin\alpha\cos I}{\tan\gamma}(\cos\alpha\tan\gamma - \sin\alpha)$$

$$+ \epsilon_2 (1+h)^2\cdot\frac{\sin^2\alpha\cos^2 I}{u\tan\gamma} \tag{5.4}$$

$$\frac{d\Omega}{dh} = \frac{C_L}{C_D}\cdot\frac{Be^{-h/\epsilon_1}}{\epsilon_1}\frac{\sin\sigma\sin\alpha}{\cos\gamma\sin\gamma\sin I} + 2\left[\epsilon_2\frac{(1+h)}{u}\right]^{1/2}\cdot\frac{\sin\alpha}{\tan\gamma}(\cos\alpha\tan\gamma - \sin\alpha)$$

$$+ \epsilon_2 (1+h)^2\cdot\frac{\sin^2\alpha\cos I}{u\tan\gamma} \tag{5.5}$$

$$\frac{dI}{dh} = \frac{C_L}{C_D}\cdot\frac{Be^{-h/\epsilon_1}}{\epsilon_1}\cdot\frac{\sin\sigma\cos\alpha}{\cos\gamma\sin\gamma} + 2\left[\epsilon_2\frac{(1+h)}{u}\right]^{1/2}\cdot\frac{\cos\alpha\sin\alpha}{\tan\gamma}\cdot$$

$$\cdot(\cos\alpha\tan\gamma-\sin\alpha) - \epsilon_2(1+h)^2\cdot\frac{\cos\alpha\sin\alpha\cos I\sin I}{u\tan\gamma} \qquad (5.6)$$

Values for the two small parameters must be calculated and substituted into Eqs (5.1) - (5.6) for each planet studied. For the Earth, the value of the first small parameter is approximately 1/900 and the value of the second small parameter is approximately 1/289 . To simplify the following analysis, it is noted that the first small parameter is approximately equal to one third of the second small parameter. Hence, a new small parameter, ϵ, is defined for Earth entry analysis as

$$\epsilon = \frac{1}{30} = \epsilon_1^{\ 1/2} = (\epsilon_2/3)^{1/2}; \quad \epsilon_1 = \epsilon^2 \quad \text{and} \quad \epsilon_2 = 3\epsilon^2 \qquad (5.7)$$

<u>The Equations of Motion for Earth Atmospheric Entry</u>. To obtain the equations governing Earth atmospheric entry, the first and second small parameters are replaced in the equations of motion by ϵ and a constant. The resulting equations are used in the derivation of solutions for Earth entry:

$$\frac{du}{dh} = \frac{-u}{(1+h)} - \frac{2uBe^{-h/\epsilon^2}}{\epsilon^2\sin\gamma}\cdot\left[1+\frac{C_L}{C_D}\tan\gamma\cos\sigma\right] - 4\epsilon\left[3u(1+h)\right]^{1/2}\cos I$$

$$- 6\epsilon^2(1+h)^2\cdot\frac{\cos\alpha\sin^2 I\sin\alpha}{\tan\gamma} \qquad (5.8)$$

$$\frac{dq}{dh} = \frac{q}{(1+h)}\left[\frac{q^2}{u} - 1\right] - \frac{C_L}{C_D}\cdot\frac{Be^{-h/\epsilon^2}}{\epsilon^2}\cdot\cos\sigma - 2\epsilon q\left[\frac{3(1+h)}{u}\right]^{1/2}\cos I$$

$$- 3\epsilon^2(1+h)^2\cdot\frac{q^3}{u}(\tan\gamma\sin\alpha\cos\alpha\sin^2 I + 1 - \sin^2 I\sin^2\alpha) \qquad (5.9)$$

$$\frac{d\alpha}{dh} = \frac{1}{(1+h)\tan\gamma} - \frac{C_L}{C_D}\cdot\frac{Be^{-h/\epsilon^2}}{\epsilon^2}\cdot\frac{\sin\sigma\sin\alpha}{\cos\gamma\sin\gamma\tan I}$$

$$- 2\epsilon\left[\frac{3(1+h)}{u}\right]^{1/2}\cdot\frac{\sin\alpha\ \cos I}{\tan\gamma}(\cos\alpha\tan\gamma - \sin\alpha)$$

$$+ 3\epsilon^2(1+h)^2\cdot\frac{\sin^2\alpha\ \cos^2 I}{u\tan\gamma} \qquad (5.10)$$

$$\frac{d\Omega}{dh} = \frac{C_L}{C_D}\cdot\frac{Be^{-h/\epsilon^2}}{\epsilon^2}\frac{\sin\sigma\sin\alpha}{\cos\gamma\sin\gamma\sin I} + 2\epsilon\left[\frac{3(1+h)}{u}\right]^{1/2}\cdot\frac{\sin\alpha}{\tan\gamma}(\cos\alpha\tan\gamma - \sin\alpha)$$

$$+ 3\epsilon^2(1+h)^2\ \frac{\sin^2\alpha\ \cos I}{u\tan\gamma} \qquad (5.11)$$

$$\frac{dI}{dh} = \frac{C_L}{C_D}\cdot\frac{Be^{-h/\epsilon^2}}{\epsilon^2}\cdot\frac{\sin\sigma\cos\alpha}{\cos\gamma\sin\gamma} + 2\epsilon\left[\frac{3(1+h)}{u}\right]^{1/2}\cdot\frac{\cos\alpha\sin\alpha}{\tan\gamma}\cdot$$

$$\cdot(\cos\alpha\tan\gamma - \sin\alpha) - 3\epsilon^2(1+h)^2\cdot\frac{\cos\alpha\sin\alpha\cos I\sin I}{u\tan\gamma} \qquad (5.12)$$

It was previously seen that the latter three of the above equations are coupled:

$$\frac{d\Omega}{dh} = \frac{\tan\alpha}{\sin I}\cdot\frac{dI}{dh} \qquad (5.13)$$

$$\frac{d\alpha}{dh} = \frac{1}{(1+h)\tan\gamma} - \frac{\tan\alpha}{\tan I}\cdot\frac{dI}{dh} \qquad (5.14)$$

Eqs (5.8) - (5.12) describe the three-dimensional, rotating planetary entry of Earth. However, the generic application of these equations to some of the other planets is not lost. As an example, consider the problem of Mars atmospheric entry. To make Eqs (5.1) - (5.6) applicable for Mars entry analysis, the first and second small parameters are calculated using Mars planetary constants. A new small parameter, ϵ, differing from Earth's by only a constant is then defined. For Mars, the first small parameter has a value of approximately 1/350 . The second small parameter is calculated from IAU defined constants to be approximately 1/218.

$$\epsilon_1 = \frac{1}{\beta r} = \frac{1}{350} \qquad \epsilon_2 = \omega^2 \frac{r_*}{g_*} = \frac{1}{218}$$

The first small parameter is approximately equal to three fifths of the second small parameter. Therefore, a unique ϵ is defined for Mars entry analysis:

$$\epsilon = \frac{1}{18.7} = (\epsilon_1)^{1/2} = (3\epsilon_2/5)^{1/2}$$

$$\text{or} \quad \epsilon_1 = \epsilon^2 \quad \text{and} \quad \epsilon_2 = \frac{5\epsilon^2}{3} \tag{5.15}$$

To adjust the equations of motion for Martian atmospheric entry, the new first and second small parameters, as a function of ϵ and a constant, are substituted into Eqs (5.1) - (5.6). Hence, the equations of motion for Mars entry will differ from those for Earth entry analysis by only a constant in each expression containing ϵ.

The Method of Matched Asymptotic Expansions

While many analytical methods have been applied to the problem of planetary atmospheric entry, one method remains relatively unexploited. This is the method of matched asymptotic expansions. From an entry vehicle's viewpoint, a planet's atmosphere forms a boundary layer of density in space. Aerodynamic forces on the vehicle change from insignificant to dominating for a relatively small change in altitude. A variety of methods, including composite expansions, multiple scales, and matched asymptotic expansions, have been utilized to solve boundary value problems. However, the method of matched asymptotic expansions is more "versitile" and "effective" than these other methods for both linear and nonlinear problems composed of partial or ordinary differential equations (Nayfeh, 1985:257,258). Past papers describe successful applications of matched asymptotic expansions to solve limited flight mechanics problems involving lifting atmospheric entry. (Busemann and others:1976, Shi and Pottsepp:1969, Shi:1971, Shi and others:1971, Willes and others:1967). Most of these efforts used two-dimensional equations of motion and assumed a spherical, non-rotating Earth model except Busemann (Busemann and others, 1976), where the three-dimensional equations of motion were employed. In the following pages, the solution for the dq/dh equation of motion for a rotating Earth is derived. To obtain this solution, the solutions for the non-rotating equations of motion are first derived from the rotating Earth equations of motion.

For a boundary layer problem, the method of matched asymptotic expansions gives two or more solutions, each valid in specific regions of the domain. These solutions have some overlap and can therefore be matched. The matching conditions allow for the construction of a composite solution which is valid over the entire domain. In this analysis, the entry problem is treated as a boundary layer problem with a solution developed away from the boundary layer, valid for exo-atmospheric flight, and a solution developed at the boundary layer, valid for atmospheric flight. These two solutions to the equations of motion can be independently applied in their respective domains. However, it is more convenient to have a solution valid for the full range of entry conditions. Hence, the inner and outer solutions are matched and a composite solution valid in both regimes is constructed.

Outer Expansions

The solution developed for the exo-atmospheric portion of the domain is called the outer solution. This solution is developed from asymptotic expansions of the equations of motion using the small parameter, ϵ. The outer solution variables are denoted by the superscript "o". Straightforward expansions of the outer solution variables are made and are as follows:

$$u^{o} = u_0 + \epsilon^1 u_1 + \epsilon^2 u_2 + \ldots.$$

$$q^{o} = q_0 + \epsilon^1 q_1 + \epsilon^2 q_2 + \ldots.$$

$$I^{o} = I_0 + \epsilon^1 I_1 + \epsilon^2 I_2 + \ldots. \qquad (5.16)$$

$$\Omega^{o} = \Omega_0 + \epsilon^1 \Omega_1 + \epsilon^2 \Omega_2 + \ldots.$$

$$\alpha^{o} = \alpha_0 + \epsilon^1 \alpha_1 + \epsilon^2 \alpha_2 + \ldots.$$

$$\gamma^{o} = \gamma_0 + \epsilon^1 \gamma_1 + \epsilon^2 \gamma_2 + \ldots.$$

The solutions for lifting atmospheric entry of a non-rotating planet are of order ϵ^0. Adding the rotating planet model to the atmospheric entry problem causes additional terms of order ϵ^1 and ϵ^2 to appear in the expansions of the equations of motion. These additional terms are relatively complex but require the solutions to be carried out to order ϵ^1 in order to account for Coriolis acceleration on the flight vehicle. To also model the usually insignificant Centripetal or Transport acceleration, solutions would have to be carried out to order ϵ^2. The solutions developed in this study are to order ϵ^0 for Eqs (5.8), (5.10), (5.11), and (5.12) and to order ϵ^1 for Eq (5.9), the dq/dh equation of motion. Solutions developed to order ϵ^1 act as a correction to the zero order solutions, a correction which accounts for the significant effects of a rotating planetary atmosphere.

In the following pages, the equations of motion are expanded for small ϵ using the straight-forward expansions given by Eq (5.16). Many of the common expressions that contain ϵ in the equations of motion are expanded in more detail in Appendix B.

The du/dh Outer Expansion. The outer expansion of the du/dh equation is given by

$$\frac{du_0}{dh} + \epsilon\frac{du_1}{dh} + \epsilon^2\frac{du_1}{dh} = \frac{-\left[u_0 + \epsilon u_1 + \epsilon^2 u_2\right]}{(1+h)}$$

$$- \frac{2}{\epsilon^2}(u_0 + \epsilon u_1 + \epsilon^2 u_2)B\exp(-h/\epsilon^2)\left[\frac{1}{\sin\gamma_0} - \epsilon\gamma_1\frac{\cos\gamma_0}{\sin^2\gamma_0}\right]\cdot$$

$$\cdot\left[1 + \frac{C_L}{C_D}\cos\sigma\left[\tan\gamma_0 + \frac{\epsilon\gamma_1}{\cos^2\gamma_0}\right]\right]$$

$$- 4\epsilon\left[3(u_0 + \epsilon u_1 + \epsilon^2 u_2)(1+h)\right]^{1/2}(\cos I_0 - \epsilon I_1\sin I_0)$$

$$- 6\epsilon^2(1+h)^2\left[(\cos\alpha_0 - \epsilon\alpha_1\sin\alpha_0)(\sin\alpha_0 + \epsilon\alpha_1\cos\alpha_0)\cdot\right.$$

$$\left.\cdot(\sin^2 I_0 + 2\epsilon I_1\sin I_0\cos I_0)\left[\frac{1}{\tan\gamma_0} + \frac{\epsilon\gamma_1}{\sin^2\gamma_0}\right]\right] + O(\epsilon^2)$$

$$\epsilon^0 \text{ terms:} \quad \frac{du_0}{dh} = \frac{-u_0}{(1+h)} \tag{5.17}$$

ϵ^1 terms: $$\frac{du_1}{dh} = \frac{-u_1}{(1+h)} - 4\left[3u_0(1+h)\right]^{1/2}\cos I_0 \tag{5.18}$$

<u>The dI/dh Outer Expansion</u>. The outer expansion of the dI/dh equation is as follows:

$$\frac{dI_0}{dh} + \epsilon\frac{dI_1}{dh} + \epsilon^2\frac{dI_1}{dh} = \frac{C_L}{C_D}\cdot\frac{Be^{-h/\epsilon^2}}{\epsilon^2}\sin\sigma(\cos\alpha_0 - \epsilon\alpha_1\sin\alpha_0)\cdot$$

$$\cdot\left[\frac{1}{\cos\gamma_0} + \epsilon\gamma_1\frac{\sin\gamma_0}{\cos^2\gamma_0}\right]\left[\frac{1}{\sin\gamma_0} - \epsilon\gamma_1\frac{\cos\gamma_0}{\sin^2\gamma_0}\right]$$

$$+ 2\epsilon\left[3(1+h)\left[\frac{1}{u_0} - \epsilon\frac{u_1}{u_0{}^2}\right]\right]^{1/2}(\cos\alpha_0 - \epsilon\alpha_1\sin\alpha_0)\cdot$$

$$\cdot(\sin\alpha_0 + \epsilon\alpha_1\cos\alpha_0)\left[\frac{1}{\tan\gamma_0} - \frac{\epsilon\gamma_1}{\sin^2\gamma_0}\right]\left[(\cos\alpha_0 - \epsilon\alpha_1\sin\alpha_0)\cdot\right.$$

$$\left.\cdot\left[\tan\gamma_0 + \frac{\epsilon\gamma_1}{\cos^2\gamma_0})\right] - (\sin\alpha_0 + \epsilon\alpha_1\cos\alpha_0)\right]$$

$$- 3\epsilon^2(1+h)^2\left[\left[\frac{1}{u_0} - \epsilon\frac{u_1}{u_0{}^2}\right]\left[\frac{1}{\tan\gamma_0} - \frac{\epsilon\gamma_1}{\sin^2\gamma_0}\right]\cdot\right.$$

$$\cdot(\sin\alpha_0 + \epsilon\alpha_1\cos\alpha_0)\cdot(\cos\alpha_0 - \epsilon\alpha_1\sin\alpha_0)(\sin I_0 + \epsilon I_1\cos I_0)\cdot$$

$$\left.\cdot(\cos I_0 - \epsilon I_1\sin I_0)\right] + O(\epsilon^2)$$

ϵ^0 terms: $$\frac{dI_0}{dh} = 0 \tag{5.19}$$

ϵ^1 terms:

$$\frac{dI_1}{dh} = 2\left[\frac{3(1+h)}{u_0}\right]^{1/2} \frac{\cos\alpha_0 \sin I_0}{\tan\gamma_0}(\cos\alpha_0 \tan\gamma_0 - \sin\alpha_0) \tag{5.20}$$

<u>The dΩ/dh Outer Expansion</u>. It has been noted that the dΩ/dh equation is a function of the dI/dh equation as well as a function of α and I. This dependence exists for the outer expansions of these equations as well. From Eq (5.13)

$$\frac{d\Omega}{dh} = \frac{\tan\alpha}{\sin I} \cdot \frac{dI}{dh}$$

The outer expansion of the dΩ/dh equation is

$$\frac{d\Omega_0}{dh} + \epsilon\frac{d\Omega_1}{dh} + \epsilon^2\frac{d\Omega_1}{dh} = \left[\frac{dI_0}{dh} + \epsilon\frac{dI_1}{dh} + \epsilon^2\frac{dI_1}{dh}\right] \cdot$$

$$\cdot\left[\tan\alpha_0 + \frac{\epsilon\alpha_1}{\cos^2\alpha_0}\right]\left[\frac{1}{\sin I_0} - \epsilon I_1 \frac{\cos I_0}{\sin^2 I_0}\right] + O(\epsilon^2)$$

$$\epsilon^0 \text{ terms:} \quad \frac{d\Omega_0}{dh} = 0 \quad \text{since} \quad \frac{dI_0}{dh} = 0 \tag{5.21}$$

$$\epsilon^1 \text{ terms:} \quad \frac{d\Omega_1}{dh} = \frac{\tan\alpha_0}{\sin I_0} \cdot \frac{dI_1}{dh}$$

$$\frac{d\Omega_1}{dh} = 2\left[\frac{3(1+h)}{u_0}\right]^{1/2} \frac{\sin\alpha_0}{\tan\gamma_0}(\cos\alpha_0 \tan\gamma_0 - \sin\alpha_0) \tag{5.22}$$

<u>The dα/dh Outer Expansion</u>. It has been noted that the dα/dh equation is a function of the dI/dh equation as well as a function of α, h, γ, and I. This dependence exists for the outer expansions of these equations as well. From

Eq (5.14)

$$\frac{d\alpha}{dh} = \frac{1}{(1+h)\tan\gamma} - \frac{\tan\alpha}{\tan I}\cdot\frac{dI}{dh}$$

The outer expansion of the $d\alpha/dh$ equation is

$$\frac{d\alpha_0}{dh} + \epsilon\frac{d\alpha_1}{dh} + \epsilon^2\frac{d\alpha_1}{dh} = \frac{1}{(1+h)}\left[\frac{1}{\tan\gamma_0} - \frac{\epsilon\gamma_1}{\sin^2\gamma_0}\right]$$

$$- \left[\frac{dI_0}{dh} + \epsilon\frac{dI_1}{dh} + \epsilon^2\frac{dI_1}{dh}\right]\left[\frac{1}{\tan I_0} - \frac{\epsilon I_1}{\sin^2 I_0}\right]\left[\tan\alpha_0 + \frac{\epsilon\alpha_1}{\cos^2\alpha_0}\right] + O(\epsilon^2)$$

$$\epsilon^0 \text{ terms:} \quad \frac{d\alpha_0}{dh} = \frac{1}{(1+h)\tan\gamma_0} \qquad \text{since } \frac{dI_0}{dh} = 0 \tag{5.23}$$

$$\epsilon^1 \text{ terms:} \quad \frac{d\alpha_1}{dh} = \frac{-\gamma_1}{(1+h)\sin^2\gamma_0} - \frac{\tan\alpha_0}{\tan I_0}\cdot\frac{dI_1}{dh}$$

$$\frac{d\alpha_1}{dh} = \frac{-\gamma_1}{(1+h)\sin^2\gamma_0}$$

$$- 2\left[\frac{(1+h)}{3u_0}\right]^{1/2}\frac{\sin\alpha_0\sin I_0}{\tan\gamma_0\tan I_0}\cdot(\cos\alpha_0\tan\gamma_0 - \sin\alpha_0) \tag{5.24}$$

The dq/dh Outer Expansion. The outer expansion of the dq/dh equation is as follows:

$$\frac{dq_0}{dh} + \epsilon\frac{dq_1}{dh} + \epsilon^2\frac{dq_1}{dh} = \frac{\left[q_0 + \epsilon q_1 + \epsilon^2 q_2\right]}{(1+h)}\left[(q_0{}^2 + 2\epsilon q_1)\cdot\right.$$

$$\left.\cdot\left[\frac{1}{u_0} - \frac{\epsilon u_1}{u_0{}^2}\right] - 1\right] - \frac{C_L}{C_D}\cdot\frac{Be^{-h/\epsilon^2}}{\epsilon^2}\cos\sigma - 2\epsilon(q_0 + \epsilon q_1 + \epsilon^2 q_2)\cdot$$

$$\cdot\left[3(1+h)\left[\frac{1}{u_0} - \frac{\epsilon u_1}{u_0^{\ 2}}\right]\right]^{1/2} \cdot (\cos I_0 - \epsilon I_1 \sin I_0)$$

$$- 3\epsilon^2 (1+h)^2 (q_0^{\ 3} + 3q_0^{\ 2}\epsilon q_1)\left[\frac{1}{u_0} - \frac{\epsilon u_1}{u_0^{\ 2}}\right]\left[\left[\tan\gamma_0 + \frac{\epsilon\gamma_1}{\cos^2\gamma_0}\right]\cdot\right.$$

$$\cdot(\sin\alpha_0 + \epsilon\alpha_1 \cos\alpha_0)(\cos\alpha_0 - \epsilon\alpha_1 \sin\alpha_0)\cdot$$

$$\cdot(\sin^2 I_0 + 2\epsilon I_1 \sin I_0 \cos I_0) + 1 - (\sin^2 I_0 + 2\epsilon I_1 \sin I_0 \cos I_0)\cdot$$

$$\left.\cdot(\sin^2\alpha_0 + 2\epsilon\alpha_1 \sin\alpha_0 \cos\alpha_0)\right] + O(\epsilon^2)$$

$$\epsilon^0 \text{ terms:} \quad \frac{dq_0}{dh} = \frac{q_0}{(1+h)}\left[\frac{q_0^{\ 2}}{u_0} - 1\right] \tag{5.25}$$

$$\epsilon^1 \text{ terms:} \quad \frac{dq_1}{dh} = \frac{q_0}{(1+h)}\left[-\frac{q_0^{\ 2}u_1}{u_0^{\ 2}} + \frac{2q_1}{u_0}\right] + \frac{q_1}{(1+h)}\left[\frac{q_0^{\ 2}}{u_0} - 1\right]$$

$$- 2q_0\left[\frac{3(1+h)}{u_0}\right]^{1/2}\cos I_0 \tag{5.26}$$

Outer Expansion Solutions

ϵ^0 Terms. The complete set of ϵ^0 term outer expansion differential equations, derived above, are

$$\frac{du_0}{dh} = \frac{-u_0}{(1+h)} \tag{5.17}$$

$$\frac{dI_0}{dh} = 0 \tag{5.19}$$

$$\frac{d\Omega_0}{dh} = 0 \tag{5.21}$$

$$\frac{d\alpha_0}{dh} = \frac{1}{(1+h)\tan\gamma_0} \tag{5.23}$$

$$\frac{dq_0}{dh} = \frac{q_0}{(1+h)}\left[\frac{q_0^{\,2}}{u_0} - 1\right] \tag{5.25}$$

Solutions to this set of differential equations are derived in Appendix C and are given below. Eq (5.30) is the outer zero order ϵ solution to the dq/dh equation of motion.

$$I_0 = C_5 \tag{5.27}$$

$$\Omega_0 = C_4 \tag{5.28}$$

$$u_0 (1+h) = C_1 \tag{5.29}$$

$$\frac{1}{q_0^{\,2}} = \frac{2(1+h)}{C_1} - C_2 (1+h)^2 \tag{5.30}$$

$$\alpha_0 = -\cos^{-1}\left[\frac{1 - C_1/(1+h)}{(1 - C_1^{\,2}C_2)^{1/2}}\right] + C_3 \qquad \text{or}$$

$$u_0 = 1 + \cos(\alpha_0 - C_3)\left[1 - C_1^{\,2}C_2\right]^{1/2} \tag{5.31}$$

<u>ϵ^1 Terms</u>. The complete set of ϵ^1 term outer expansion differential equations, derived above, are

$$\frac{du_1}{dh} = \frac{-u_1}{(1+h)} - 4\left[3u_0 (1+h)\right]^{1/2}\cdot\cos I_0 \tag{5.18}$$

$$\frac{dI_1}{dh} = 2\left[\frac{3(1+h)}{u_0}\right]^{1/2}\cdot\frac{\cos\alpha_0 \sin I_0}{\tan\gamma_0}(\cos\alpha_0 \tan\gamma_0 - \sin\alpha_0) \tag{5.20}$$

$$\frac{d\Omega_1}{dh} = \frac{\tan\alpha_0}{\sin I_0} \cdot \frac{dI_1}{dh} \tag{5.22}$$

or
$$\frac{d\Omega_1}{dh} = 2\left[\frac{3(1+h)}{u_0}\right]^{1/2} \frac{\sin\alpha_0}{\tan\gamma_0}(\cos\alpha_0 \tan\gamma_0 - \sin\alpha_0)$$

$$\frac{d\alpha_1}{dh} = \frac{-\gamma_1}{(1+h)\sin^2\gamma_0} - \frac{\tan\alpha_0}{\tan I_0} \cdot \frac{dI_1}{dh} \tag{5.24}$$

or

$$\frac{d\alpha_1}{dh} = \frac{-\gamma_1}{(1+h)\sin^2\gamma_0} - 2\left[\frac{3(1+h)}{u_0}\right]^{1/2} \cdot \frac{\sin\alpha_0 \cos I_0}{\tan\gamma_0}(\cos\alpha_0 \tan\gamma_0 - \sin\alpha_0)$$

$$\frac{dq_1}{dh} = \frac{q_0}{(1+h)}\left[-\frac{q_0^{\,2} u_1}{u_0^{\,2}} + \frac{2q_1}{u_0}\right] + \frac{q_1}{(1+h)}\left[\frac{q_0^{\,2}}{u_0} - 1\right]$$

$$- 2q_0\left[\frac{3(1+h)}{u_0}\right]^{1/2} \cos I_0 \tag{5.26}$$

A complete solution to the dq/dh equation requires that the solutions to both Eqs (5.18) and (5.26) be found. Solutions to these differential equations are derived in Appendix C and are given below. Eqs (5.32) and (5.33) are the solutions to the order ϵ terms in the outer solutions of the du/dh and dq/dh equations, respectively.

$$u_1 = \frac{C_6}{(1+h)} - 2(3C_1)^{1/2}(1+h)\cos C_5 \tag{5.32}$$

$$q_1 = -\frac{C_{13}}{C_{16}} - \frac{C_{14}}{(C_{16})^2} - h \cdot \frac{C_{14}}{C_{16}} - C_{20}\exp\left[-C_{16}h\right] \tag{5.33}$$

where C_{13}, C_{14}, and C_{16} are functions of C_1 and C_2, and C_{20} is a constant of integration. Expressions for these constants are given in Appendix C.

Order ϵ Outer Solution for the dq/dh Equation of Motion. The order ϵ outer solution for the cosine of the flight path angle is given by the substitution of Eqs (5.30) and (5.33) into Eq (5.16).

$$q^o = q_0 + \epsilon q_1 + O(\epsilon^2)$$

$$q^o = \left[\frac{2(1+h)}{C_1} - C_2(1+h)^2\right]^{-1/2}$$

$$- \epsilon\left[\frac{C_{13}}{C_{16}} - \frac{C_{14}}{(C_{16})^2} - h\cdot\frac{C_{14}}{C_{16}} - C_{20}\exp\left[-C_{16}h\right]\right] \tag{5.34}$$

Inner Expansions

The solution developed for the atmospheric portion of the domain is called the inner solution. This solution is developed from asymptotic expansions of the equations of motion using the small parameter, ϵ. The outer solution variables are denoted by the superscript "i". Straightforward expansions of the inner solution variables are made and are given below.

$$u^i = u_0 + \epsilon^1 u_1 + \epsilon^2 u_2 + \ldots.$$

$$q^i = q_0 + \epsilon^1 q_1 + \epsilon^2 q_2 + \ldots.$$

$$I^i = I_0 + \epsilon^1 I_1 + \epsilon^2 I_2 + \ldots. \tag{5.35}$$

$$\Omega^i = \Omega_0 + \epsilon^1 \Omega_1 + \epsilon^2 \Omega_2 + \ldots.$$

$$\alpha^i = \alpha_0 + \epsilon^1 \alpha_1 + \epsilon^2 \alpha_2 + \ldots.$$

$$\gamma^i = \gamma_0 + \epsilon^1 \gamma_1 + \epsilon^2 \gamma_2 + \ldots.$$

To derive the inner expansions, a new independent variable is required so as to force the equations of motion to focus on the boundary layer region. ξ, a magnified version of h, is the inner expansion independent variable. Hence, ξ is defined as the magnified non-dimensional altitude and it is given by

$$\xi = h/\epsilon^2 \quad \text{or} \quad h = \epsilon^2 \xi \tag{5.36}$$

Therefore $\frac{dh}{d\xi} = \epsilon^2$ and by the chain rule,

$$\frac{dy}{d\xi} = \frac{dy}{dh}\cdot\frac{dh}{d\xi} = \epsilon^2 \frac{dy}{dh} \tag{5.37}$$

The equations of motion for Earth atmospheric entry are now transformed from functions of h, represented by Eqs (5.8) - (5.12), to functions of the magnified non-dimensional altitude, ξ.

$$\frac{du^i}{d\xi} = \frac{-\epsilon^2 u^i}{(1+\epsilon^2\xi)} - \frac{2u^i Be^{-\xi}}{\sin\gamma^i}\cdot\left[1 + \frac{C_L}{C_D}\tan\gamma^i\cos\sigma\right] \tag{5.38}$$

$$- 4\epsilon^3\left[3u^i(1 + \epsilon^2\xi)\right]^{1/2}\cos I^i - 6\epsilon^4(1+\epsilon^2\xi)^2\cdot\frac{\cos\alpha^i\sin^2 I^i\sin\alpha^i}{\tan\gamma^i}$$

$$\frac{dq^i}{d\xi} = \frac{\epsilon^2 q^i}{(1+\epsilon^2\xi)}\left[\frac{q^{i^2}}{u^i} - 1\right] - \frac{C_L}{C_D}Be^{-\xi}\cos\sigma - 2\epsilon^3 q^i\left[\frac{3(1+\epsilon^2\xi)}{u^i}\right]^{1/2}\cos I^i$$

$$- 3\epsilon^4(1+\epsilon^2\xi)^2\cdot\frac{q^{i^3}}{u^i}(\tan\gamma^i\sin\alpha^i\cos\alpha^i\sin^2 I^i$$

$$+ 1 - \sin^2 I^i\sin^2\alpha^i) \qquad (5.39)$$

$$\frac{d\alpha^i}{d\xi} = \frac{\epsilon^2}{(1+\epsilon^2\xi)\tan\gamma^i} - \frac{C_L}{C_D}\cdot Be^{-\xi}\cdot\frac{\sin\sigma\sin\alpha^i}{\cos\gamma^i\sin\gamma^i\tan I^i}$$

$$- 2\epsilon^3\left[\frac{3(1+\epsilon^2\xi)}{u^i}\right]^{1/2}\cdot\frac{\sin\alpha^i\cos I^i}{\tan\gamma^i}(\cos\alpha^i\tan\gamma^i - \sin\alpha^i)$$

$$+ 3\epsilon^4(1+\epsilon^2\xi)^2\cdot\frac{\sin^2\alpha^i\cos^2 I^i}{u^i\tan\gamma^i} \qquad (5.40)$$

$$\frac{d\Omega^i}{d\xi} = \frac{C_L}{C_D}\cdot Be^{-\xi}\cdot\frac{\sin\sigma^i\sin\alpha^i}{\cos\gamma^i\sin\gamma^i\sin I^i} + 2\epsilon^3\left[\frac{3(1+\epsilon^2\xi)}{u^i}\right]^{1/2}\cdot \qquad (5.41)$$

$$\cdot\frac{\sin\alpha^i}{\tan\gamma^i}(\cos\alpha^i\tan\gamma^i - \sin\alpha^i) + 3\epsilon^4(1+\epsilon^2\xi)^2\cdot\frac{\sin^2\alpha^i\cos I^i}{u^i\tan\gamma^i}$$

$$\frac{dI^i}{d\xi} = \frac{C_L}{C_D}\cdot Be^{-\xi}\cdot\frac{\sin\sigma\cos\alpha^i}{\cos\gamma^i\sin\gamma^i} + 2\epsilon^3\left[\frac{3(1+\epsilon^2\xi)}{u^i}\right]^{1/2}\cos\alpha^i\cdot$$

$$\cdot\frac{\sin\alpha^i}{\tan\gamma^i}(\cos\alpha^i\tan\gamma^i - \sin\alpha^i)$$

$$- 3\epsilon^4(1+\epsilon^2\xi)^2\cdot\frac{\cos\alpha^i\sin\alpha^i\cos I^i\sin I^i}{u^i\tan\gamma^i} \qquad (5.42)$$

The latter three of the above equations are coupled.

$$\frac{d\Omega^i}{d\xi} = \frac{\tan\alpha^i}{\sin I^i}\cdot\frac{dI^i}{d\xi} \qquad (5.43)$$

$$\frac{d\alpha^i}{d\xi} = \frac{\epsilon^2}{(1+\epsilon^2\xi)\tan\gamma^i} - \frac{\tan\alpha^i}{\tan I^i}\cdot\frac{dI^i}{d\xi} \tag{5.44}$$

In the following pages, these equations of motion are expanded for small ϵ using the straight-forward expansions of Eq (5.35). Many of the common expressions that contain ϵ are expanded in detail in Appendix B.

The $du/d\xi$ Equation Expansion. The inner expansion of the du/dh equation is given by

$$\frac{du_0}{d\xi} + \epsilon\frac{du_1}{d\xi} + \epsilon^2\frac{du_1}{d\xi} = -\epsilon^2\frac{\left[u_0+\epsilon u_1+\epsilon^2 u_2\right]}{(1+\epsilon^2\xi)}$$

$$- 2(u_0+\epsilon u_1+\epsilon^2 u_2)Be^{-\xi}\cdot\left[\frac{1}{\sin\gamma_0} - \epsilon\gamma_1\frac{\cos\gamma_0}{\sin^2\gamma_0}\right]\cdot$$

$$\cdot\left[1 + \frac{C_L}{C_D}\cos\sigma\left[\tan\gamma_0 + \frac{\epsilon\gamma_1}{\cos^2\gamma_0}\right]\right]$$

$$- 4\epsilon^3\left[\frac{3(u_0+\epsilon u_1+\epsilon^2 u_2)(1+\epsilon^2\xi)}{1}\right]^{1/2}(\cos I_0 - \epsilon I_1\sin I_0)$$

$$- 6\epsilon^4(1+\epsilon^2\xi)^2\left[(\cos\alpha_0 - \epsilon\alpha_1\sin\alpha_0)(\sin\alpha_0 + \epsilon\alpha_1\cos\alpha_0)\cdot\right.$$

$$\left.\cdot(\sin^2 I_0 + 2\epsilon^3 I_1\sin I_0\cos I_0)\left[\frac{1}{\tan\gamma_0} - \frac{\epsilon\gamma_1}{\sin^2\gamma_0}\right]\right] + O(\epsilon^2)$$

This immediately reduces to

$$\frac{du_0}{d\xi} + \epsilon\frac{du_1}{d\xi} + O(\epsilon^2) = - 2(u_0+\epsilon u_1)Be^{-\xi}\cdot\left[\frac{1}{\sin\gamma_0} - \epsilon\gamma_1\frac{\cos\gamma_0}{\sin^2\gamma_0}\right.$$

$$+ \frac{C_L}{C_D}\cdot\frac{\cos\sigma}{\cos\gamma_0} - \epsilon\gamma_1 \frac{C_L}{C_D}\cdot\frac{\cos\sigma}{\sin\gamma_0} + \frac{C_L}{C_D}\cdot\frac{\epsilon\gamma_1\cos\sigma}{\sin\gamma_0\cos^2\gamma_0}\Bigg] + O(\epsilon^2)$$

$$\epsilon^0 \text{ terms:} \quad \frac{du_0}{d\xi} = -2u_0 Be^{-\xi}\cdot\left[\frac{1}{\sin\gamma_0} + \frac{C_L}{C_D}\cdot\frac{\cos\sigma}{\cos\gamma_0}\right] \tag{5.45}$$

$$\epsilon^1 \text{ terms:} \quad \frac{du_1}{d\xi} = 2u_0 Be^{-\xi}\cdot\Bigg[\gamma_1\frac{\cos\gamma_0}{\sin^2\gamma} + \gamma_1\frac{C_L}{C_D}\cdot\frac{\cos\sigma}{\cos\gamma_0} \tag{5.46}$$

$$- \frac{C_L}{C_D}\cdot\frac{\gamma_1\cos\sigma}{\sin\gamma_0\cos^2\gamma_0}\Bigg] - 2u_1 Be^{-\xi}\cdot\left[\frac{1}{\sin\gamma_0} + \frac{C_L}{C_D}\cdot\frac{\cos\sigma}{\cos\gamma_0}\right] + O(\epsilon^2)$$

<u>The $dI/d\xi$ Equation Expansion</u>. The inner expansion of the $dI/d\xi$ equation is

$$\frac{dI_0}{d\xi} + \epsilon\frac{dI_1}{d\xi} + O(\epsilon^2) = \frac{C_L}{C_D}\cdot Be^{-\xi}\sin\sigma(\cos\alpha_0 - \epsilon\alpha_1\sin\alpha_0)\cdot$$

$$\cdot\left[\frac{1}{\cos\gamma_0} + \epsilon\gamma_1\frac{\sin\gamma_0}{\cos^2\gamma_0}\right]\left[\frac{1}{\sin\gamma_0} - \epsilon\gamma_1\frac{\cos\gamma_0}{\sin^2\gamma_0}\right]$$

$$+ 2\epsilon^3\left[3(1+\epsilon^2\xi)\left[\frac{1}{u_0} - \epsilon\frac{u_1}{u_0{}^2}\right]\right]^{1/2}(\cos\alpha_0 - \epsilon\alpha_1\sin\alpha_0)\cdot$$

$$\cdot(\sin\alpha_0 + \epsilon\alpha_1\cos\alpha_0)\left[\frac{1}{\tan\gamma_0} - \frac{\epsilon\gamma_1}{\sin^2\gamma_0}\right]\Bigg[\Big[(\cos\alpha_0 - \epsilon\alpha_1\sin\alpha_0)\cdot$$

$$\cdot\left[\tan\gamma_0 + \frac{\epsilon\gamma_1}{\sin^2\gamma_0}\right] - (\sin\alpha_0 + \epsilon\alpha_1\cos\alpha_0)\Bigg]$$

$$- 3\epsilon^4(1+\epsilon^2\xi)^2\left[\left[\frac{1}{u_0} - \epsilon\frac{u_1}{u_0{}^2}\right]\left[\frac{1}{\tan\gamma_0} - \frac{\epsilon\gamma_1}{\sin^2\gamma_0}\right]\cdot\right.$$

$$\cdot(\sin\alpha_0 + \epsilon\alpha_1\cos\alpha_0)(\cos\alpha_0 - \epsilon\alpha_1\sin\alpha_0)\cdot$$

$$\cdot(\sin I_0 + \epsilon I_1 \cos I_0)(\cos I_0 - \epsilon I_1 \sin I_0)\Big] + O(\epsilon^2)$$

This equation reduces to

$$\frac{dI_0}{d\xi} + \epsilon\frac{dI_1}{d\xi} + O(\epsilon^2) = \frac{C_L}{C_D}\cdot Be^{-\xi}\sin\sigma(\cos\alpha_0 - \epsilon\alpha_1\sin\alpha_0)\cdot$$

$$\cdot\left[\frac{1}{\cos\gamma_0} + \epsilon\gamma_1\frac{\sin\gamma_0}{\cos^2\gamma_0}\right]\left[\frac{1}{\sin\gamma_0} - \epsilon\gamma_1\frac{\cos\gamma_0}{\sin^2\gamma_0}\right] + O(\epsilon^2)$$

ϵ^0 terms: $$\frac{dI_0}{d\xi} = \frac{C_L}{C_D}\cdot Be^{-\xi}\sin\sigma\cdot\frac{\cos\alpha_0}{\cos\gamma_0\sin\gamma_0} \tag{5.47}$$

ϵ^1 terms: (5.48)

$$\frac{dI_1}{d\xi} = \frac{C_L}{C_D}\cdot Be^{-\xi}\cdot\sin\sigma\left[-\alpha_1\frac{\sin\alpha_0}{\cos\gamma_0\sin\gamma_0} + \gamma_1\frac{\cos\alpha_0}{\cos^2\gamma_0} - \gamma_1\frac{\cos\alpha_0}{\sin^2\gamma_0}\right]$$

<u>The $d\Omega/d\xi$ Equation Expansion</u>. It has been noted that the $d\Omega/d\xi$ equation is a function of the $dI/d\xi$ equation as well as a function of α and I. This dependence exists for the inner expansions of these equations as well.

$$\frac{d\Omega^i}{d\xi} = \frac{\tan\alpha^i}{\sin I^i}\cdot\frac{dI^i}{d\xi}$$

The inner expansion of the $d\Omega/d\xi$ equation is

$$\frac{d\Omega_0}{d\xi} + \epsilon\frac{d\Omega_1}{d\xi} + \epsilon^2\frac{d\Omega_1}{d\xi} = \left[\frac{dI_0}{d\xi} + \epsilon\frac{dI_1}{d\xi} + \epsilon^2\frac{dI_1}{d\xi}\right]\cdot$$

$$\cdot\left[\tan\alpha_0 + \frac{\epsilon\alpha_1}{\cos^2\alpha_0}\right]\left[\frac{1}{\sin I_0} - \epsilon I_1\frac{\cos I_0}{\sin^2 I_0}\right] + O(\epsilon^2)$$

With substitutions, the $d\Omega/d\xi$ equation expansion becomes

$$\frac{d\Omega_0}{d\xi} + \epsilon\frac{d\Omega_1}{d\xi} + \epsilon^2\frac{d\Omega_1}{d\xi} = \frac{C_L}{C_D}\cdot Be^{-\xi}\cdot \sin\sigma(\cos\alpha_0 - \epsilon\alpha_1\sin\alpha_0)\cdot$$

$$\cdot\left[\frac{1}{\cos\gamma_0} + \epsilon\gamma_1\frac{\sin\gamma_0}{\cos^2\gamma_0}\right]\left[\frac{1}{\sin\gamma_0} - \epsilon\gamma_1\frac{\cos\gamma_0}{\sin^2\gamma_0}\right]\cdot$$

$$\cdot\left[\tan\alpha_0 + \frac{\epsilon\alpha_1}{\cos^2\alpha_0}\right]\left[\frac{1}{\sin I_0} - \epsilon I_1\frac{\cos I_0}{\sin^2 I_0}\right] + O(\epsilon^2)$$

$$\epsilon^0 \text{ terms:} \quad \frac{d\Omega_0}{d\xi} = \frac{C_L}{C_D}\cdot Be^{-\xi}\cdot\frac{\sin\sigma\sin\alpha_0}{\cos\gamma_0\sin\gamma_0\sin I_0} \tag{5.49}$$

$$\epsilon^1 \text{ terms:} \quad \frac{d\Omega_1}{d\xi} = \frac{C_L}{C_D}\cdot Be^{-\xi}\sin\sigma\left[\gamma_1\frac{\sin\alpha_0}{\cos^2\gamma_0\sin I_0} - \gamma_1\frac{\sin\alpha_0}{\sin^2\gamma_0\sin I_0}\right.$$

$$\left. - I_1\frac{\sin\alpha_0\cos I_0}{\cos\gamma_0\sin\gamma_0\sin^2 I_0} + \alpha_1\frac{\cos\alpha_0}{\cos\gamma_0\sin\gamma_0\sin I_0}\right] \tag{5.50}$$

The $d\alpha/d\xi$ Equation Expansion. It has been noted that the $d\alpha/d\xi$ equation is a function of the $dI/d\xi$ equation as well as a function of α, ξ, γ, and I. This dependence exists for the outer expansions of these equations as well. From Eq (5.40)

$$\frac{d\alpha^i}{d\xi} = \frac{1}{(1+\epsilon^2\xi)\tan\gamma^i} - \frac{\tan\alpha^i}{\tan I^i}\cdot\frac{dI^i}{d\xi}$$

The inner expansion of the $d\alpha/d\xi$ equation is

$$\frac{d\alpha_0}{d\xi} + \epsilon\frac{d\alpha_1}{d\xi} + \epsilon^2\frac{d\alpha_1}{d\xi} = \frac{\epsilon^2}{(1+\epsilon^2\xi)}\left[\frac{1}{\tan\gamma_0} - \frac{\epsilon\gamma_1}{\sin^2\gamma_0}\right]$$

$$- \left[\frac{dI_0}{d\xi} + \epsilon\frac{dI_1}{d\xi} + \epsilon^2\frac{dI_1}{d\xi}\right]\left[\frac{1}{\tan I_0} - \frac{\epsilon I_1}{\sin^2 I_0}\right]\cdot$$

$$\cdot\left[\tan\alpha_0 + \frac{\epsilon\alpha_1}{\cos^2\alpha_0}\right] + O(\epsilon^2)$$

With substitution, the following expression is obtained:

$$\frac{d\alpha_0}{d\xi} + \epsilon\frac{d\alpha_1}{d\xi} + \epsilon^2\frac{d\alpha_1}{d\xi} = \frac{\epsilon^2}{(1+\epsilon^2\xi)}\left[\frac{1}{\tan\gamma_0} - \frac{\epsilon\gamma_1}{\sin^2\gamma_0}\right]$$

$$- \frac{C_L}{C_D}\cdot Be^{-\xi}\sin\sigma(\cos\alpha_0 - \epsilon\alpha_1\sin\alpha_0)\left[\frac{1}{\cos\gamma_0} + \epsilon\gamma_1\frac{\sin\gamma_0}{\cos^2\gamma_0}\right]\cdot$$

$$\cdot\left[\frac{1}{\sin\gamma_0} - \epsilon\gamma_1\frac{\cos\gamma_0}{\sin^2\gamma_0}\right]\left[\frac{1}{\tan I_0} - \frac{\epsilon I_1}{\sin^2 I_0}\right]\cdot$$

$$\cdot\left[\tan\alpha_0 + \frac{\epsilon\alpha_1}{\cos^2\alpha_0}\right] + O(\epsilon^2)$$

$$\epsilon^0 \text{ terms:} \quad \frac{d\alpha_0}{d\xi} = -\frac{C_L}{C_D}\cdot Be^{-\xi}\cdot\frac{\sin\sigma\sin\alpha_0}{\tan I_0\sin\gamma_0\cos I_0} \tag{5.51}$$

$$\epsilon^1 \text{ terms:} \quad \frac{d\alpha_1}{d\xi} = -\frac{C_L}{C_D}\cdot Be^{-\xi}\sin\sigma\left[\gamma_1\frac{\sin\alpha_0}{\cos^2\gamma_0\tan I_0} - \gamma_1\frac{\sin\alpha_0}{\sin^2\gamma_0\tan I_0}\right.$$

$$\left.- I_1\frac{\sin\alpha_0}{\cos\gamma_0\sin\gamma_0\sin^2 I_0} + \alpha_1\frac{\cos\alpha_0}{\cos\gamma_0\sin\gamma_0\tan I_0}\right] \tag{5.52}$$

<u>The dq/dξ Equation Expansion</u>. The outer expansion of the $dq/d\xi$ equation is

$$\frac{dq_0}{d\xi} + \epsilon\frac{dq_1}{d\xi} + \epsilon^2\frac{dq_1}{d\xi} = \epsilon^2\frac{\left[q_0 + \epsilon q_1 + \epsilon^2 q_2\right]}{(1+\epsilon^2\xi)}\left[\left[q_0{}^2 + 2\epsilon q_1 + O(\epsilon^2)\right]\cdot\right.$$

$$\cdot\left[\frac{1}{u_0} - \frac{\epsilon u_1}{u_0{}^2}\right] - 1\Bigg] - \frac{C_L}{C_D}\cdot Be^{-\xi}\cos\sigma - 2\epsilon^3(q_0 + \epsilon q_1 + \epsilon^2 q_2)\cdot$$

$$\cdot\left[3(1+\epsilon^2\xi)\left[\frac{1}{u_0} - \frac{\epsilon u_1}{u_0{}^2}\right]\right]^{1/2}(\cos I_0 - \epsilon I_1 \sin I_0)$$

$$- 3\epsilon^4(1+\epsilon^2\xi)^2(q_0{}^3 + 3q_0{}^2\epsilon q_1)\left[\frac{1}{u_0} - \frac{\epsilon u_1}{u_0{}^2}\right]\Bigg[\left[\tan\gamma_0 + \frac{\epsilon\gamma_1}{\cos^2\gamma_0}\right]\cdot$$

$$\cdot(\sin\alpha_0 + \epsilon\alpha_1\cos\alpha_0)(\cos\alpha_0 - \epsilon\alpha_1\sin\alpha_0)\cdot$$

$$\cdot(\sin^2 I_0 + 2\epsilon I_1 \sin I_0 \cos I_0) + 1 - (\sin^2 I_0 + 2\epsilon I_1 \sin I_0 \cos I_0)\cdot$$

$$\cdot(\sin^2\alpha_0 + 2\epsilon\alpha_1\sin\alpha_0\cos\alpha_0)\Bigg] + O(\epsilon^2)$$

$$\epsilon^0 \text{ terms:} \quad \frac{dq_0}{d\xi} = -\frac{C_L}{C_D}\cdot Be^{-\xi}\cos\sigma \tag{5.53}$$

$$\epsilon^1 \text{ terms:} \quad \frac{dq_1}{d\xi} = 0 \tag{5.54}$$

Inner Expansion Solutions

ϵ^0 Terms. The complete set of ϵ^0 term inner expansion differential equations, derived above, are as follows:

$$\frac{du_0}{d\xi} = -2u_0 Be^{-\xi}\cdot\left[\frac{1}{\sin\gamma_0} + \frac{C_L}{C_D}\cdot\frac{\cos\sigma}{\cos\gamma_0}\right] \tag{5.45}$$

$$\frac{dI_0}{d\xi} = \frac{C_L}{C_D}\cdot Be^{-\xi}\cdot\frac{\sin\sigma\cos\alpha_0}{\cos\gamma_0\sin\gamma_0} \tag{5.47}$$

$$\frac{d\Omega_0}{d\xi} = \frac{C_L}{C_D}\cdot Be^{-\xi}\cdot\frac{\sin\sigma\sin\alpha_0}{\cos\gamma_0\sin\gamma_0\sin I_0} \tag{5.49}$$

$$\frac{d\alpha_0}{d\xi} = -\frac{C_L}{C_D}\cdot Be^{-\xi}\cdot\frac{\sin\sigma\sin\alpha_0}{\tan I_0\sin\gamma_0\cos I_0} \tag{5.51}$$

$$\frac{dq_0}{d\xi} = -\frac{C_L}{C_D}\cdot Be^{-\xi}\cos\sigma \tag{5.53}$$

Solutions to this set of differential equations are derived in Appendix D and are given below. Eq (5.55) is the inner zero order ϵ solution to the dq/dh equation of motion.

$$q_0 = \frac{C_L}{C_D}\cdot Be^{-\xi}\cos\sigma + K_1 \tag{5.55}$$

$$u_0 = K_3 q_0^{\,2}\exp\left[\frac{-2C_D\gamma_0}{C_L\cos\sigma}\right] \tag{5.56}$$

$$\sin\alpha_0\sin I_0 = \sin K_4 \tag{5.57}$$

$$\cos\alpha_0 = \cos K_4\cos(K_5 - \Omega_0) \tag{5.58}$$

$$\cos I_0 = \cos K_4\cos\left[\tan\sigma\cdot\log\left[\tan(\frac{\pi}{4} + \frac{\gamma_0}{2})\right] + K_6\right] \tag{5.59}$$

<u>ϵ^1 Terms</u>. The complete set of ϵ^1 term inner expansion differential equations, derived above, are as follows:

$$\frac{du_1}{d\xi} = 2u_0 Be^{-\xi}\cdot\left[\gamma_1\frac{\cos\gamma_0}{\sin^2\gamma} + \gamma_1\frac{C_L}{C_D}\cdot\frac{\cos\sigma}{\cos\gamma_0} - \frac{C_L}{C_D}\cdot\frac{\gamma_1\cos\sigma}{\sin\gamma_0\cos^2\gamma_0}\right]$$

$$- 2u_1 Be^{-\xi}\cdot\left[\frac{1}{\sin\gamma_0} + \frac{C_L}{C_D}\cdot\frac{\cos\sigma}{\cos\gamma_0}\right] + O(\epsilon^2) \tag{5.46}$$

$$\frac{dI_1}{d\xi} = \frac{C_L}{C_D} \cdot Be^{-\xi} \sin\sigma \cdot$$

$$\cdot \left[-\alpha_1 \frac{\sin\alpha_0}{\cos\gamma_0 \sin\gamma_0} + \gamma_1 \frac{\cos\alpha_0}{\cos^2\gamma_0} - \gamma_1 \frac{\cos\alpha_0}{\sin^2\gamma_0} \right] \tag{5.48}$$

$$\frac{d\Omega_1}{d\xi} = \frac{\tan\alpha_0}{\sin I_0} \cdot \frac{dI_1}{d\xi} \tag{5.50}$$

or
$$\frac{d\Omega_1}{d\xi} = \frac{C_L}{C_D} \cdot Be^{-\xi} \sin\sigma \left[\gamma_1 \frac{\sin\alpha_0}{\cos^2\gamma_0 \sin I_0} - \gamma_1 \frac{\sin\alpha_0}{\sin^2\gamma_0 \sin I_0} \right.$$
$$\left. - I_1 \frac{\sin\alpha_0 \cos I_0}{\cos\gamma_0 \sin\gamma_0 \sin^2 I_0} + \alpha_1 \frac{\cos\alpha_0}{\cos\gamma_0 \sin\gamma_0 \sin I_0} \right]$$

$$\frac{d\alpha_1}{d\xi} = - \frac{\tan\alpha_0}{\tan I_0} \cdot \frac{dI_1}{d\xi} \tag{5.52}$$

or
$$\frac{d\alpha_1}{d\xi} = - \frac{C_L}{C_D} \cdot Be^{-\xi} \sin\sigma \left[\gamma_1 \frac{\sin\alpha_0}{\cos^2\gamma_0 \tan I_0} - \gamma_1 \frac{\sin\alpha_0}{\sin^2\gamma_0 \tan I_0} \right.$$
$$\left. - I_1 \frac{\sin\alpha_0}{\cos\gamma_0 \sin\gamma_0 \sin^2 I_0} + \alpha_1 \frac{\cos\alpha_0}{\cos\gamma_0 \sin\gamma_0 \tan I_0} \right]$$

$$\frac{dq_1}{d\xi} = 0 \tag{5.54}$$

A complete solution to the dq/dh equation requires that only the solution to Eq (5.54) be found from this set of equations. The solution to Eq (5.54) is trivial; the constant of integration is defined as K_7.

$$q_1 = K_7 \tag{5.60}$$

Order ϵ Inner Solution for the dq/dh Equation of Motion. The order ϵ inner solution to the cosine of the flight path angle is given by the substitution of Eqs (5.55) and (5.60) into Eq (5.35).

$$q^i = q_0 + \epsilon q_1 + O(\epsilon^2) \tag{5.35}$$

$$q^i = \frac{C_L}{C_D} \cdot Be^{-\xi} \cos\sigma + K_1 + \epsilon K_7 \tag{5.61}$$

At this point, the inner and outer solutions to the equations of motion have been derived to order ϵ^0. These are the solutions to the atmospheric entry for a non-rotating Earth. In addition, the inner and outer solutions to the dq/dh equation of motion have been derived to order ϵ. These latter two solutions include the Coriolis acceleration on the flight vehicle and therefore account for the significant effects of a rotating Earth. In order to create composite solutions valid for both the inner and outer regions, matching is performed.

Matching Zero Order ϵ Solutions

One of the fundamental rules of the technique of matched asymptotic expansions is that the outer expansion of the inner expansion solution is equal to the inner expansion of the outer expansion solution (Nayfeh, 1981:277,278). This condition allows matching of the solutions, reducing the number of unknowns in the solution equations. On the following pages the inner expansions are taken of the ϵ^0 outer expansion solutions. Next, the outer expansions are taken of the ϵ^0 inner expansion solutions. Finally, the derived expansions of expansions are matched and the inner solution constants of integration are expressed in terms of the outer solution constants of integration.

Inner Expansions of Order ϵ^0 Outer Expansion Solutions. From Eq (5.29), the one term outer expansion for u is

$$(u_0)^o = C_1/(1+h)$$

Noting that $\xi = h/\epsilon^2$ or $h = \epsilon^2\xi$ and rewriting this equation in terms of the inner variable, ξ, gives

$$(u_0)^o = C_1/(1+\epsilon^2\xi)$$

Expanding for small ϵ gives the inner expansion of the outer expansion of the zero order ϵ term for u.

$$\left[(u_0)^o\right]^i = C_1 \tag{5.62}$$

From Eq (5.30), the one term outer expansion for q is

$$(q_0)^o = \left[C_1/\left[2(1 + h) - C_1C_2(1 + h)^2\right]\right]^{1/2}$$

Rewriting this equation in terms of the inner variable

$$(q_0)^o = \left[C_1/\left[2(1 + \epsilon^2\xi) - C_1C_2(1 + \epsilon^2\xi)^2\right]\right]^{1/2}$$

Expanding for small ϵ gives the inner expansion of the outer expansion of the zero order ϵ term for q.

$$\left[(q_0)^o\right]^i = \left[C_1/\left[2 + 2\epsilon^2\xi - C_1C_2 - C_1C_2\epsilon^4\xi - 2C_1C_2\epsilon^2\xi\right]\right]^{1/2}$$

$$\left[(q_0)^o\right]^i = \left[C_1/(2 - C_1C_2)\right]^{1/2} \tag{5.63}$$

From Eq (5.28), the one term outer expansion for Ω is

$$(\Omega_0)^o = C_4$$

Rewriting this equation in terms of the inner variable simply gives

$$(\Omega_0)^o = C_4$$

Expanding for small ϵ gives the inner expansion of the outer expansion of the zero order ϵ term for Ω.

$$\left[(\Omega_0)^o\right]^i = C_4 \tag{5.64}$$

From Eq (5.27), the one term outer expansion for I is

$$(I_0)^o = C_5$$

Rewriting this equation in terms of the inner variable simply gives

$$(I_0)^o = C_5$$

Expanding for small ϵ gives the inner expansion of the outer expansion of the zero order ϵ term for I.

$$\left[(I_0)^o\right]^i = C_5 \tag{5.65}$$

From Eq (5.31), the one term outer expansion for α is

$$(\alpha_0)^o = -\cos^{-1}\left[\frac{1 - C_1/(1+h)}{(1 - C_1{}^2 C_2)^{1/2}}\right] + C_3$$

Rewriting this equation in terms of the inner variable:

$$(\alpha_0)^o = -\cos^{-1}\left[\frac{1 - C_1/(1+\epsilon^2\xi)}{(1 - C_1{}^2 C_2)^{1/2}}\right] + C_3$$

Expanding for small ϵ gives the inner expansion of the outer expansion of the zero order ϵ term for α.

$$\left[(\alpha_0)^o\right]^i = -\cos^{-1}\left[\frac{1 - C_1}{(1 - C_1{}^2 C_2)^{1/2}}\right] + C_3 \tag{5.66}$$

Outer Expansions of the ϵ^0 Inner Expansion Solutions.

From Eq (5.55), the one term inner expansion for q is

$$(q_0)^i = \frac{C_L}{C_D}\cdot Be^{-\xi}\cos\sigma + K_1$$

Noting that $\xi = h/\epsilon^2$ or $h = \epsilon^2\xi$ and rewriting this equation in terms of the outer variable, h, gives

$$(q_0)^i = \frac{C_L}{C_D}\cdot Be^{-h/\epsilon^2}\cdot\cos\sigma + K_1$$

Expanding for small ϵ gives the outer expansion of the inner expansion of the zero order ϵ term for q.

$$\left[(q_0)^i\right]^o = K_1 \tag{5.67}$$

and hence $$\left[(\gamma_0)^i\right]^o = \cos^{-1} K_1 \tag{5.68}$$

From Eq (5.56), the one term inner expansion for u is

$$(u_0)^i = K_3\left[(q_0)^i\right]^2 \exp\left[\frac{-2\gamma_0}{\cos\sigma C_L/C_D}\right]$$

Rewritten in terms of the outer variable

$$(u_0)^i = K_3\left[(q_0)^i\right]^2 \exp\left[\frac{-2C_D\cos^{-1}(q_0)^i}{C_L\cos\sigma}\right]$$

Expanding for small ϵ gives the outer expansion of the inner expansion solution of the zero order ϵ term for u.

$$\left[(u_0)^i\right]^o = K_3 K_1^{\,2} \exp\left[\frac{-2\cos^{-1}K_1}{\cos\sigma C_L/C_D}\right] \tag{5.69}$$

From Eq (5.59), the one term inner expansion for I is

$$(I_0)^i = \cos^{-1}\left[\cos K_4 \cos\left[\tan\sigma\cdot\log\left[\tan\left[\frac{\pi}{4} + \frac{(\gamma_0)^i}{2}\right]\right] + K_6\right]\right]$$

Rewritten in terms of the outer variable

$$(I_0)^i = \cos^{-1}\left[\cos K_4 \cos\left[\tan\sigma\cdot\log\left[\tan\left[\frac{\pi}{4} + \frac{(\gamma_0)^i}{2}\right]\right] + K_6\right]\right]$$

Expanding for small ϵ gives the outer expansion of the inner expansion of the zero order ϵ term for q.

$$\left[(I_0)^i\right]^o = \cos^{-1}\left[\cos K_4 \cos\left[\tan\sigma \cdot \log\left[\tan\left[\frac{\pi}{4} + \frac{\cos^{-1}K_1}{2}\right]\right] + K_6\right]\right] \tag{5.70}$$

From Eq (5.57), the one term inner expansion for α is

$$(\alpha_0)^i = \sin^{-1}\left[\frac{\sin K_4}{\sin(I_0)^i}\right]$$

Rewritten in terms of the outer variable:

$$(\alpha_0)^i = \sin^{-1}\left[\frac{\sin K_4}{\sin(I_0)^i}\right]$$

Expanding for small ϵ gives the outer expansion of the inner expansion of the zero order ϵ term for α.

$$\left[(\alpha_0)^i\right]^o = \sin^{-1}\left[\sin K_4 / \sin\left[(I_0)^i\right]^o\right] \tag{5.71}$$

where $\left[(I_0)^i\right]^o$ was given by Eq (5.65)

From Eq (5.58), the one term inner expansion for Ω is

$$(\Omega_0)^i = -\cos^{-1}\left[\frac{\cos(\alpha_0)^i}{\cos K_4}\right] + K_5$$

Rewritten in terms of the outer variable:

$$(\Omega_0)^i = -\cos^{-1}\left[\frac{\cos(\alpha_0)^i}{\cos K_4}\right] + K_5$$

Expanding for small ϵ gives the outer expansion of the inner expansion of the zero order ϵ term for Ω.

$$\left[(\Omega_0)^i\right]^o = -\cos^{-1}\left[\cos\left[(\alpha_0)^i\right]^o / \cos K_4\right] + K_5 \qquad (5.72)$$

where $\left[(\alpha_0)^i\right]^o$ was given by Eq (5.66)

$\underline{\epsilon^0 \text{ Term Matching}}$. As previously discussed, the outer expansion of the inner expansion solution is equal to the inner expansion of the outer expansion solution. Mathematically,

$$\left[(x)^i\right]^o = \left[(x)^o\right]^i \qquad (5.73)$$

This is sometimes expressed (Lagerstrom, 1972:90) as

$$\lim_{out}\left[\lim_{in} x\right] = \lim_{in}\left[\lim_{out} x\right]$$

where $x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \ldots\ldots$

The expressions derived for the outer expansion of the inner expansion solution and for the inner expansion of the outer expansion solution can be matched for q, u, α, I, and Ω.

Matching the Speed Ratio Equations. From Eq (5.73)

$$\left[(u_0)^i\right]^o = \left[(u_0)^o\right]^i$$

Therefore, from Eqs (5.62) and (5.69)

$$C_1 = K_3 K_1^{\,2} \exp\left[\frac{-2\cos^{-1} K_1}{\cos\sigma C_L/C_D}\right] \tag{5.74}$$

<u>Matching the Flight Path Angle Equations</u>.

$$\left[(q_0)^i\right]^o = \left[(q_0)^o\right]^i$$

Setting Eq (5.63) equal to Eq (5.57) gives

$$K_1 = \left[C_1/(2 - C_1 C_2)\right]^{1/2} \tag{5.75}$$

<u>Matching the Inclination Angle Equations</u>.

$$\left[(I_0)^i\right]^o = \left[(I_0)^o\right]^i$$

Equating Eqs (5.65) and (5.70) gives

$$C_5 = \cos^{-1}\left[\cos K_4 \cos\left[\tan\sigma \cdot \right.\right.$$

$$\left.\left. \cdot \log\left[\tan\left[\frac{\pi}{4} + \frac{\cos^{-1} K_1}{2}\right]\right] + K_6\right]\right] \tag{5.76}$$

<u>Matching the Argument of Latitude at Epoch Equations</u>.

$$\left[(\alpha_0)^i\right]^o = \left[(\alpha_0)^o\right]^i$$

Equating Eqs (5.66) and (5.71) gives

$$- \cos^{-1}\left[\frac{1 - C_1}{(1 - C_1{}^2 C_2)^{1/2}}\right] + C_3 = \sin^{-1}\left[\sin K_4 / \sin\left[(I_0)^i\right]^0\right]$$

and since $\left[(I_0)^i\right]^o = \left[(I_0)^o\right]^i = C_5$ then

$$- \cos^{-1}\left[\frac{1 - C_1}{(1 - C_1{}^2 C_2)^{1/2}}\right] + C_3 = \sin^{-1}\left[\frac{\sin K_4}{\sin C_5}\right] \tag{5.77}$$

<u>Matching the Longitude of the Ascending Node Equations</u>.

$$\left[(\Omega_0)^i\right]^o = \left[(\Omega_0)^o\right]^i$$

Setting Eq (5.64) equal to Eq (5.72)

$$C_4 = -\cos^{-1}\left[\cos\left[(\alpha_0)^i\right]^o / \cos K_4\right] + K_5$$

$$C_4 = -\cos^{-1}\left[\cos\left[- \cos^{-1}\left[\frac{1 - C_1}{(1 - C_1{}^2 C_2)^{1/2}}\right] + C_3\right] / \cos K_4\right] + K_5 \tag{5.78}$$

The inner expansion solutions, valid in the domain of h near the boundary layer where aerodynamic forces are dominate, have been matched with the outer expansion solutions, valid in the domain of h away from the boundary layer. The results of the matchings are used to reduce the number of unknowns in the solution equations. The constants of the outer expansions are determined first, often from initial conditions, in a planetary entry problem. Hence, it is desirable to express the constants of the inner expansions, K, in terms of the constants of the outer expansions, C. From Eq (5.75)

$$K_1 = \left[C_1/(2 - C_1C_2)\right]^{1/2} \tag{5.79}$$

From Eqs (5.74) and (5.75)

$$K_3 = (2 - C_1C_2)\exp\left[\frac{-2\cos^{-1}\left[C_1/(2 - C_1C_2)\right]^{1/2}}{\cos\sigma \cdot C_L/C_D}\right] \tag{5.80}$$

From Eq (5.77)

$$K_4 = \sin^{-1}\left[\sin C_5 \cdot \sin\left[- \cos^{-1}\left[\frac{1 - C_1}{(1 - C_1{}^2C_2)^{1/2}}\right] + C_3\right]\right] \tag{5.81}$$

From Eq (5.78)

$$K_5 = C_4 + \cos^{-1}\left[\cos\left[- \cos^{-1}\left[\frac{1 - C_1}{(1 - C_1{}^2C_2)^{1/2}}\right]\right.\right.$$

$$\left.\left. + C_3\right]/\cos K_4\right] \tag{5.82}$$

From Eq (5.76)

$$K_6 = \cos^{-1}\left[\frac{\cos C_5}{\cos K_4}\right]$$

$$- \tan\sigma \cdot \log\left[\tan\left[\frac{\pi}{4} + \frac{\cos^{-1}K_1}{2}\right]\right] \tag{5.83}$$

With Eqs (5.79) - (5.83) the inner solutions can be written in terms of the constants, C.

Zero Order ϵ Solutions to the Equations of Motion

A solution good for the entire range of h is desired for the convenience of not having to apply the two sets of solutions to one entry problem. The composite solution, valid everywhere in the domain, combines the inner and outer solutions that were valid only in certain overlapping regions of the non-dimensional altitude domain. This composite solution is given in the following equation by the method of matched asymptotic expansions.

$$y^c = y^o + y^i - \left[(y)^o\right]^i = y^o + y^i - \left[(y)^i\right]^o \tag{5.84}$$

The composite solution is therefore given by the sum of the inner and outer solutions and the difference between this result and any expression common to both the inner and outer solutions. In addition, the inner expansion of the composite solution is equal to the inner solution, and the outer expansion of the composite solution is equal to the outer solution. Mathematically,

$$\left[(y)^c\right]^i = y^i \quad \text{and} \quad \left[(y)^c\right]^o = y^o \tag{5.85}$$

Speed Ratio Composite Solution.

$$u^c = (u)^o + (u)^i - \left[(u)^o\right]^i = (u)^o + (u)^i - \left[(u)^i\right]^o$$

Combining Eqs (5.29), (5.56), and (5.62) gives the composite solution for the speed ratio.

$$u^c = C_1\left[\frac{1}{(1+h)} - 1\right] + K_3\left[\frac{C_L}{C_D}\cdot Be^{-h/\epsilon^2}\cdot \cos\sigma + K_1\right]^2 \cdot$$

$$\cdot \exp\left[-2\cos^{-1}\left[\frac{C_L}{C_D}\cdot Be^{-h/\epsilon^2}\cdot \cos\sigma + K_1\right] \Big/ \frac{C_L}{C_D}\cos\sigma\right] \qquad (5.86)$$

Flight Path Angle Composite Solution.

$$q^c = (q)^o + (q)^i - \left[(q)^o\right]^i = (q)^o + (q)^i - \left[(q)^i\right]^o$$

Combining Eqs (5.30), (5.55), and (5.67) gives a composite solution for the cosine of the flight path angle.

$$q^c = \left[C_1 / \left[2(1+h) - C_1 C_2 (1+h)^2\right]\right]^{1/2}$$

$$+ \frac{C_L}{C_D}\cdot Be^{-h/\epsilon^2}\cdot \cos\sigma \qquad (5.87)$$

Inclination Angle Composite Solution.

$$I^c = (I)^o + (I)^i - \left[(I)^o\right]^i = (I)^o + (I)^i - \left[(I)^i\right]^o$$

Combining Eqs (5.27), (5.59), and (5.70) gives a composite solution for the orbital inclination angle.

$$I^c = \cos^{-1}\Bigg(\cos K_4 \cos\Bigg[\tan\sigma \cdot \qquad (5.88)$$

$$\cdot \log\left(\tan\left[\frac{\pi}{4} + \frac{1}{2}\cos^{-1}\left(\frac{C_L}{C_D}Be^{-h/\epsilon^2}\cdot \cos\sigma + K_1\right)\right]\right) + K_6\Bigg]\Bigg)$$

Longitude of the Ascending Node Composite Solution.

$$\alpha^c = (\alpha)^o + (\alpha)^i - \left[(\alpha)^o\right]^i = (\alpha)^o + (\alpha)^i - \left[(\alpha)^i\right]^o$$

Combining Eqs (5.31), (5.59), and (5.66) gives a composite solution for the longitude of the ascending node.

$$\alpha^c = -\cos^{-1}\left[\frac{1 - C_1/(1+h)}{(1 - C_1{}^2 C_2)^{1/2}}\right]$$

$$+ \sin^{-1}\left[\sin K_4/\cos^{-1}\left(\cos K_4 \cos\left[\tan\sigma\cdot\log\left(\tan\left[\frac{\pi}{4}\right.\right.\right.\right.\right.$$

$$\left.\left.\left.\left.\left.+ \frac{1}{2}\left(\frac{C_L}{C_D}Be^{-h/\epsilon^2}\cdot\cos\sigma + K_1\right)\right]\right)\right] + K_6\right)\right]$$

$$+ \cos^{-1}\left[\frac{1 - C_1}{(1 - C_1{}^2 C_2)^{1/2}}\right] \tag{5.89}$$

Argument of Latitude at Epoch Composite Solution.

$$\Omega^c = (\Omega)^o + (\Omega)^i - \left[(\Omega)^o\right]^i = (\Omega)^o + (\Omega)^i - \left[(\Omega)^i\right]^o$$

Combining Eqs (5.28), (5.58), and (5.72) gives a composite solution for the argument of latitude at epoch.

$$\Omega^c = -\cos^{-1}\left[\frac{\cos(\alpha^i)}{\cos K_4}\right] + K_5$$

$$\Omega^c = K_5 - \cos^{-1}\left[\cos\left(\sin^{-1}\left[\sin K_4/\sin(I^c)\right]\right)/\cos K_4\right] \tag{5.90}$$

For Eqs (5.86) - (5.90), the K_i constants are given by Eqs (5.79) - (5.83). Hence, solutions to the five equations of motion have been developed as functions of C_1, C_2, C_3, C_4, and C_5.

Matching ϵ Order Solutions to the dq/dh Equation of Motion

An inner expansion of the outer expansion solution is undertaken as well as an outer expansion of the inner expansion solution. Matching is performed by equating these expansions of expansion solutions.

Inner Expansion of the Outer Expansion Solution. The outer expansion solution for the dq/dh equation of motion was derived to order ϵ and given by Eq (5.34).

$$q^{\circ} = \left[\frac{2(1+h)}{C_1} - C_2(1+h)^2\right]^{-1/2}$$

$$- \epsilon\left[\frac{C_{13}}{C_{16}} - \frac{C_{14}}{(C_{16})^2} - h\cdot\frac{C_{14}}{C_{16}} - C_{20}\exp\left[-C_{16}h\right]\right]$$

Noting that $\xi = h/\epsilon^2$ and $h = \epsilon^2\xi$ and rewriting the above equation in terms of the inner variable, ξ, gives

$$(q)^{\circ} = \left[C_1/\left[2(1 + \epsilon^2\xi) - C_1C_2(1 + \epsilon^2\xi)^2\right]\right]^{1/2}$$

$$- \epsilon\left[\frac{C_{13}}{C_{16}} - \frac{C_{14}}{(C_{16})^2} - \epsilon^2\xi\cdot\frac{C_{14}}{C_{16}} - C_{20}\exp\left[-C_{16}\epsilon^2\xi\right]\right] \qquad (5.91)$$

Expanding for small ϵ gives the inner expansion of the outer expansion solution to order ϵ.

$$\left[(q)^{o}\right]^{i} = \left[C_1/2 - C_1C_2)\right]^{1/2}$$

$$- \epsilon\left[\frac{C_{13}}{C_{16}} - \frac{C_{14}}{(C_{16})^2} - C_{20}\right] \tag{5.92}$$

<u>Outer Expansion of the Inner Expansion Solution</u>. The inner expansion solution for the dq/dh equation of motion was derived to order ϵ and given by Eq (5.61).

$$(q)^{i} = \frac{C_L}{C_D} \cdot Be^{-\xi}\cos\sigma + K_1 + \epsilon K_7$$

Rewriting this equation in terms of the outer variable, h, and expanding for small ϵ gives the outer expansion of the inner expansion solution for q.

$$\left[(q)^{i}\right]^{o} = K_1 + \epsilon K_7 \tag{5.93}$$

<u>Matching</u>. The outer expansion of the inner expansion solution is equated to the inner expansion of the outer solution expansion.

$$\left[(q)^{i}\right]^{o} = \left[(q)^{o}\right]^{i}$$

$$K_1 + \epsilon K_7 = \left[C_1/2 - C_1C_2)\right]^{1/2} - \epsilon\left[\frac{C_{13}}{C_{16}} - \frac{C_{14}}{(C_{16})^2} - C_{20}\right]$$

Subtracting Eq (5.79) from this expression and dividing by ϵ gives

$$K_7 = -\frac{C_{13}}{C_{16}} - \frac{C_{14}}{(C_{16})^2} - C_{20} \tag{5.94}$$

ϵ Order Composite Solution to the dq/dh Equation of Motion

The composite solution to the dq/dh equation can be easily constructed:

$$q^c = (q)^o + (q)^i - \left[(q)^i\right]^o = (q)^o + (q)^i - \left[(q)^o\right]^i$$

$$q^c = \left[C_1/\left[2(1+h) - C_1 C_2 (1+h)^2\right]\right]^{1/2} + \frac{C_L}{C_D}\cdot Be^{-h/\epsilon^2}\cdot \cos\sigma$$

$$- \epsilon\left[\frac{C_{13}}{C_{16}} - \frac{C_{14}}{(C_{16})^2} - h\cdot\frac{C_{14}}{C_{16}} - C_{20}\exp\left[-C_{16}h\right]\right] \tag{5.95}$$

where C_{13}, C_{14}, and C_{16} are given by the following relations derived in Appendix C:

$$C_{13} = 2\cos(C_5)\left[\frac{1}{C_1 C_{11}} - 1\right]\cdot\left[\frac{3}{C_1 C_{11}}\right]^{1/2} - \frac{3C_6}{(C_1)^2 (C_{11})^{3/2}} \tag{5.96}$$

$$C_{14} = \frac{3C_6 C_{12}}{(C_1)^2 (C_{11})^{5/2}} + 2\cos(C_5)\left[1 - 3\frac{C_{12}}{C_{11}}\right]\cdot\left[\frac{3}{C_1{}^3 C_{11}{}^3}\right]^{1/2} \tag{5.97}$$

$$C_{16} = \frac{2}{C_1 (C_{11})^{1/2}} + \frac{1}{C_1 C_{11}} - 1 \tag{5.98}$$

where

$$C_{11} = \left[\frac{2}{C_1} - C_2\right] \quad \text{and} \quad C_{12} = \left[\frac{1}{C_1} - C_2\right] \tag{5.99}$$

and where C_6 and C_{20} are constants of integration.

Hence, a solution for q has been found as a function of two constants of integration, C_6 and C_{20}, and the constants C_{13}, C_{14}, and C_{16}, which are dependent upon C_1 and C_2.

Summary

For convenience, a summation of the derivation results using matched asymptotic expansions are presented.

Solutions to the Non-Rotating Earth Equations of Motion.

Treatment of atmospheric entry as a boundary layer problem allowed for the application of the method of matched asymptotic expansions. In this section, the rotating Earth equations of motion were asymptotically expanded using a small parameter that is a function of the planet's inverse atmospheric scale height, radius, and rotation rate. Zero order solutions to these equations were derived. These solutions were of two types, inner solutions, valid close to the Earth, and outer solutions, valid far from the Earth. The domains of these two sets of solutions overlapped. Hence, direct matchings of the expansions of the expansion solutions were accomplished. This produced equations that related the inner solution constants, K_i, to the outer solution constants, C_i. Composite solutions, valid for the entire atmospheric entry domain of non-dimensional altitude, were constructed using these relations. These composite solutions are solutions to the non-rotating Earth equations of motion for three-dimensional, lifting atmospheric entry.

The outer solutions were given by

$$I_0 = C_5 \tag{5.27}$$

$$\Omega_0 = C_4 \tag{5.28}$$

$$u_0 (1+h) = C_1 \tag{5.29}$$

$$\frac{1}{q_0{}^2} = \frac{2(1+h)}{C_1} - C_2 (1+h)^2 \tag{5.30}$$

$$\alpha_0 = -\cos^{-1}\left[\frac{1 - C_1/(1+h)}{(1 - C_1{}^2 C_2)^{1/2}}\right] + C_3 \qquad \text{or}$$

$$u_0 = 1 + \cos(\alpha_0 - C_3)\left[1 - C_1{}^2 C_2\right]^{1/2} \tag{5.31}$$

The inner solutions were given by

$$q_0 = \frac{C_L}{C_D}\cdot Be^{-\xi}\cos\sigma + K_1 \tag{5.55}$$

$$u_0 = K_3 q_0{}^2 \exp\left[\frac{-2C_D\gamma_0}{C_L\cos\sigma}\right] \tag{5.56}$$

$$\sin\alpha_0 \sin I_0 = \sin K_4 \tag{5.57}$$

$$\cos\alpha_0 = \cos K_4 \cos(K_5 - \Omega_0) \tag{5.58}$$

$$\cos I_0 = \cos K_4 \cos\left[\tan\sigma\cdot\log\left[\tan(\frac{\pi}{4} + \frac{\gamma_0}{2})\right] + K_6\right] \tag{5.59}$$

Matching the expansions of these solution sets produced the following relationships between the outer solution constants, C_i, and the inner solution constants, K_i.

$$K_1 = \left[C_1/(2 - C_1C_2)\right]^{1/2} \tag{5.79}$$

$$K_3 = (2 - C_1C_2)\exp\left[\frac{-2\cos^{-1}\left[C_1/(2 - C_1C_2)\right]^{1/2}}{\cos\sigma \cdot C_L/C_D}\right] \tag{5.80}$$

$$K_4 = \sin^{-1}\left[\sin C_5 \cdot \sin\left[- \cos^{-1}\left[\frac{1 - C_1}{(1 - C_1^{\,2}C_2)^{1/2}}\right] + C_3\right]\right] \tag{5.81}$$

$$K_5 = C_4 + \cos^{-1}\left[\cos\left[- \cos^{-1}\left[\frac{1 - C_1}{(1 - C_1^{\,2}C_2)^{1/2}}\right]\right.\right.$$

$$\left.\left. + C_3\right]/\cos K_4\right] \tag{5.82}$$

$$K_6 = \cos^{-1}\left[\frac{\cos C_5}{\cos K_4}\right]$$

$$- \tan\sigma \cdot \log\left[\tan\left[\frac{\pi}{4} + \frac{\cos^{-1}K_1}{2}\right]\right] \tag{5.83}$$

The equations given above are for the three-dimensional atmospheric entry of a non-rotating Earth. These solutions were derived from zero order terms in the expansions of the rotating Earth equations of motion.

The composite solutions for the three-dimensional atmospheric entry of a non-rotating Earth were given by

$$u^c = C_1\left[\frac{1}{(1+h)} - 1\right] + K_3\left[\frac{C_L}{C_D}\cdot Be^{-h/\epsilon^2}\cdot\cos\sigma + K_1\right]^2\cdot$$

$$\cdot\exp\left[-2\cos^{-1}\left[\frac{C_L}{C_D}\cdot Be^{-h/\epsilon^2}\cdot\cos\sigma + K_1\right]/\frac{C_L}{C_D}\cos\sigma\right] \tag{5.86}$$

$$q^c = \left[C_1/\left[2(1+h) - C_1C_2(1+h)^2\right]\right]^{1/2}$$

$$+ \frac{C_L}{C_D}\cdot Be^{-h/\epsilon^2}\cdot\cos\sigma \tag{5.87}$$

$$I^c = \cos^{-1}\left(\cos K_4\cos\left[\tan\sigma\cdot\right.\right. \tag{5.88}$$

$$\left.\left.\cdot\log\left(\tan\left[\frac{\pi}{4} + \frac{1}{2}\cos^{-1}\left(\frac{C_L}{C_D}Be^{-h/\epsilon^2}\cdot\cos\sigma + K_1\right)\right]\right) + K_6\right]\right)$$

$$\alpha^c = -\cos^{-1}\left[\frac{1 - C_1/(1+h)}{(1 - C_1{}^2C_2)^{1/2}}\right]$$

$$+ \sin^{-1}\left[\sin K_4/\cos^{-1}\left(\cos K_4\cos\left[\tan\sigma\cdot\log\left(\tan\left[\frac{\pi}{4}\right.\right.\right.\right.\right.$$

$$\left.\left.\left.\left.\left. + \frac{1}{2}\cos^{-1}\left(\frac{C_L}{C_D}Be^{-h/\epsilon^2}\cdot\cos\sigma + K_1\right)\right]\right) + K_6\right]\right)\right]$$

$$+ \cos^{-1}\left[\frac{1 - C_1}{(1 - C_1{}^2C_2)^{1/2}}\right] \tag{5.89}$$

$$\Omega^c = K_5 - \cos^{-1}\left[\cos\left(\sin^{-1}\left[\sin K_4/\sin(I^c)\right]\right)/\cos K_4\right] \tag{5.90}$$

<u>Solution to the dq/dh Rotating Earth Equation of Motion</u>.

To accurately model Earth atmospheric entry, a solution to the dq/dh equation that included rotating Earth effects was required. This solution was derived in the form of order ϵ inner and outer solutions to the asymptotically expanded dq/dh equation of motion for a rotating Earth. Matching was performed between these solutions and a composite solution to the dq/dh equation of motion for the three-dimensional atmospheric entry of a rotating Earth was constructed.

The outer solution to the rotating Earth dq/dh equation of motion was given by

$$q^{o} = \left[\frac{2(1+h)}{C_1} - C_2 (1+h)^2\right]^{-1/2}$$

$$- \epsilon\left[\frac{C_{13}}{C_{16}} - \frac{C_{14}}{(C_{16})^2} - h\cdot\frac{C_{14}}{C_{16}} - C_{20}\exp\left[-C_{16}h\right]\right] \tag{5.34}$$

The inner solution to the rotating Earth dq/dh equation of motion was given by

$$q^{i} = \frac{C_L}{C_D}\cdot Be^{-\xi}\cos\sigma + K_1 + \epsilon K_7 \tag{5.61}$$

The composite solution to the dq/dh equation of motion is of order ϵ and hence includes Earth rotational terms. The composite solution to the dq/dh equation of motion for the three-dimensional atmospheric entry of a rotating Earth was given by

$$q^{c} = \left[C_1/\left[2(1+h) - C_1 C_2 (1+h)^2\right]\right]^{1/2} + \frac{C_L}{C_D}\cdot Be^{-h/\epsilon^2}\cdot\cos\sigma$$

$$- \epsilon \left[\frac{C_{13}}{C_{16}} - \frac{C_{14}}{(C_{16})^2} - h \cdot \frac{C_{14}}{C_{16}} - C_{20} \exp\left[-C_{16} h \right] \right] \qquad (5.95)$$

where C_{13}, C_{14}, and C_{16} were given by the following relations derived in Appendix C:

$$C_{13} = 2\cos(C_5) \left[\frac{1}{C_1 C_{11}} - 1 \right] \cdot \left[\frac{3}{C_1 C_{11}} \right]^{1/2} - \frac{3C_6}{(C_1)^2 (C_{11})^{3/2}}$$

$$C_{14} = \frac{3C_6 C_{12}}{(C_1)^2 (C_{11})^{5/2}} + 2\cos(C_5) \left[1 - 3\frac{C_{12}}{C_{11}} \right] \cdot \left[\frac{3}{C_1{}^3 C_{11}{}^3} \right]^{1/2}$$

$$C_{16} = \frac{2}{C_1 (C_{11})^{1/2}} + \frac{1}{C_1 C_{11}} - 1$$

where

$$C_{11} = \left[\frac{2}{C_1} - C_2 \right] \quad \text{and} \quad C_{12} = \left[\frac{1}{C_1} - C_2 \right]$$

and where C_6 and C_{20} are constants of integration.

VI. Trajectory States of Validity for the Non-Rotating Earth Equations of Motion

Introduction

In Section IV, the rotating planet terms in each of the five equations of motion for Earth atmospheric entry were examined. The dq/dh equation of motion for a non-rotating Earth was found to be never valid for the investigated ranges of inclination angle and speed. Because of this, a solution was developed for the dq/dh equation of motion for the three-dimensional lifting atmospheric entry of a rotating, spherical Earth. This was accomplished in Section V by treating atmospheric entry as a boundary layer problem and applying the method of matched asymptotic expansions.

In Section IV, trajectory states were found to exist where some of the non-rotating Earth equations of motion are valid for a rotating Earth. The $d\Omega/dh$, $d\alpha/dh$, and dI/dh non-rotating equations of motion were all found to be valid for the same entry trajectory states. Other, independent trajectory states were found to exist where the du/dh non-rotating equation of motion is valid for a rotating Earth.

In this section, these trajectory states are examined in more detail for the du/dh and dI/dh equations of motion. Plots of the Rotate term solutions to these equations are generated for a large range of realistic values of u, h, α, γ, and I. Trends in these solutions are discussed and the non-existence of overlapping trajectory state solutions between the du/dh and dI/dh Rotate equations is verified. A few methods to estimate solutions to the du/dh and dI/dh Rotate equations are also presented.

Solutions to the du/dh Rotate Equation. In Section IV it was found that the rotating planet terms in the du/dh equation of motion are

$$\left.\frac{du}{dh}\right|_{Rotate} = -4\left[\epsilon_2 u(1+h)\right]^{1/2}\cos I$$

$$-2\epsilon_2 (1+h)^2 \frac{\cos\alpha \sin^2 I \sin\alpha}{\tan\gamma} \qquad (6.1)$$

Setting this equation equal to zero and solving for flight path angle gives solutions for trajectory states where the rotating Earth terms in the du/dh equation of motion are zero.

$$\left.\gamma\right|_{Rotate = 0} = \tan^{-1}\left[\frac{-C\cos\alpha\sin\alpha\sin^2 I}{2\cos I}\right] \qquad (6.2)$$

where $$C = \left[\frac{(1+h)^3}{u}\epsilon_2\right]^{1/2} \qquad (6.3)$$

Solutions to the dΩ/dh, dα/dh, and dI/dh Rotate Equations. In Section IV, the solutions for the Rotate term expressions were found to be identical for the dα/dh, dΩ/dh, and dI/dh equations of motion. This result simplifies the search for trajectory states where the non-rotating Earth equations of motion are valid for a rotating Earth. Instead of detailed examination of the solutions to all three of the $d\alpha/dh$, $d\Omega/dh$, and dI/dh Rotate term expressions, examination of only one of them is required. In this section the solutions to the dI/dh Rotate term expression are examined. From Eq (4.13)

$$\left.\frac{dI}{dh}\right|_{Rotate} = 2\left[\epsilon_2 \frac{(1+h)}{u}\right]^{1/2} \cdot \frac{\cos\alpha \sin I}{\tan\gamma}(\cos\alpha\tan\gamma - \sin\alpha)$$

$$- \epsilon_2 (1+h)^2 \cdot \frac{\cos\alpha\sin\alpha\cos I\sin I}{u\tan\gamma} \tag{6.4}$$

Setting this equation equal to zero and solving for flight path angle gives solutions for trajectory states where the rotating Earth terms in the dI/dh equation of motion are zero.

$$\gamma\Big|_{Rotate\ =\ 0} = \tan^{-1}\left[\frac{\sin\alpha + C\cos I\sin\alpha}{2\cos\alpha}\right] \tag{6.5}$$

<u>Non-Existence of Overlapping Solutions to the du/dh and dI/dh Rotate Equations</u>

To verify the non-existence of overlapping solutions to the du/dh Rotate and dI/dh Rotate equations, Eqs (6.2) and (6.5) are equated.

$$\tan^{-1}\left[\frac{-C\cos\alpha\sin\alpha\sin^2 I}{2\cos I}\right] = \tan^{-1}\left[\frac{\sin\alpha + C\cos I\sin\alpha}{2\cos\alpha}\right]$$

Simplifying

$$-C\cos^2\alpha\sin\alpha\sin^2 I = \cos I\sin\alpha + C\cos^2 I\sin\alpha \tag{6.6}$$

A trivial solution to this expression is given by $\sin\alpha = 0$. The non-trivial solutions to Eq (6.6) are found by dividing by $\sin\alpha$ and then solving for $\cos\alpha$.

$$\cos\alpha = \left[\frac{-\cos I}{C} - \cos^2 I\right]^{1/2} / \sin I \qquad (6.7)$$

where $\quad C = \left[\frac{(1+h)^3}{u}\epsilon_2\right]^{1/2}$

For real solutions to exist for Eq (6.7), the following condition must be true:

$$\frac{\cos I}{C} + \cos^2 I < 0 \qquad (6.8)$$

Since values of non-dimensional altitude and speed ratio are never negative, C always has a positive value. The term cosI is never negative for prograde entry trajectories. Hence, the condition given by Eq (6.8) is never satisfied and overlapping solutions to the du/dh and dI/dh Rotate equations do not exist.

Graphical Trajectory State Examination

In the following pages, the trajectory states which occur when the Rotate equations are zero are examined in more detail. This is equivalent to finding trajectory states where the non-rotating Earth equations of motion are equivalent to the rotating Earth equations of motion for lifting atmospheric entry. Solutions for γ are examined which result from placing a wide range of possible combinations of u, h, α, and I into the Rotate solution equations, given by Eqs (6.2) and (6.5), for the du/dh and dI/dh equations of motion. These solutions give trajectory states for which the non-rotating equations of motion are perfectly valid for rotating Earth entry analysis. This validity is dependent upon the physical characteristics of the planet under consideration, namely ϵ_2, and the type of trajectory being analyzed; validity is not directly dependent on any of the vehicle's physical characteristics. Determination of the validity of the non-rotating equations of motion for planets other than Earth could be undertaken in a similar manner as the analysis given in this section.

A computer program was developed to solve Eqs (6.2) and (6.5) for a large range of u, h, α, and I. Plots of families of curves of γ versus I are presented for various ranges of α and for realistic combinations of u and h. Appendix E presents three example Earth atmospheric entry trajectories to give an indication of what the dimensionless variables u and h are in relation to more conventional entry trajectory parameters such as Mach, altitude, and velocity. Most current and planned lifting entry vehicles have trajectories which fall within the ranges of u and h given by these examples. The realistic combinations of u and h discussed in the following pages are derived from liberal estimation of their ranges in these example trajectories. Values of u selected for investigation ranged from circular orbital velocities to low supersonic speeds where terminal maneuvers, such as landing approaches, are usually initiated. Realistic values of non-dimensional altitude corresponding to these values of speed ratio were used in this analysis. Values of argument of latitude at epoch and orbital inclination angle had ranges of $0.0 \leq \alpha \leq 360$ degrees and $0.5 \leq I \leq 75$ degrees, where most atmospheric entry occurs. Because of the large number of plots needed to display data trends, some of the plots which are discussed in this section are contained in Appendix F. In all the plots referred to in this section, α, the argument of latitude at epoch, is referred to as "Alpha". All values of γ, α, and I given in these plots are in units of degrees.

Trajectory States of Validity for the du/dh Non-Rotating Earth Equation of Motion. Trajectory states are presented below where the non-rotating Earth du/dh equation of motion is valid. Trends in these solutions to Eq (6.2) are discussed.

Typical Solution Observations. Figure 13 is a typical plot of solutions to Eq (6.2) of flight path angle versus orbital inclination angle. For constant, non-zero values of argument of latitude at epoch, γ decreases as I increases. Values of flight path angle also decrease in this plot as α increases. In addition, as α increases for a constant I, the change in γ decreases, especially for large inclination angles.

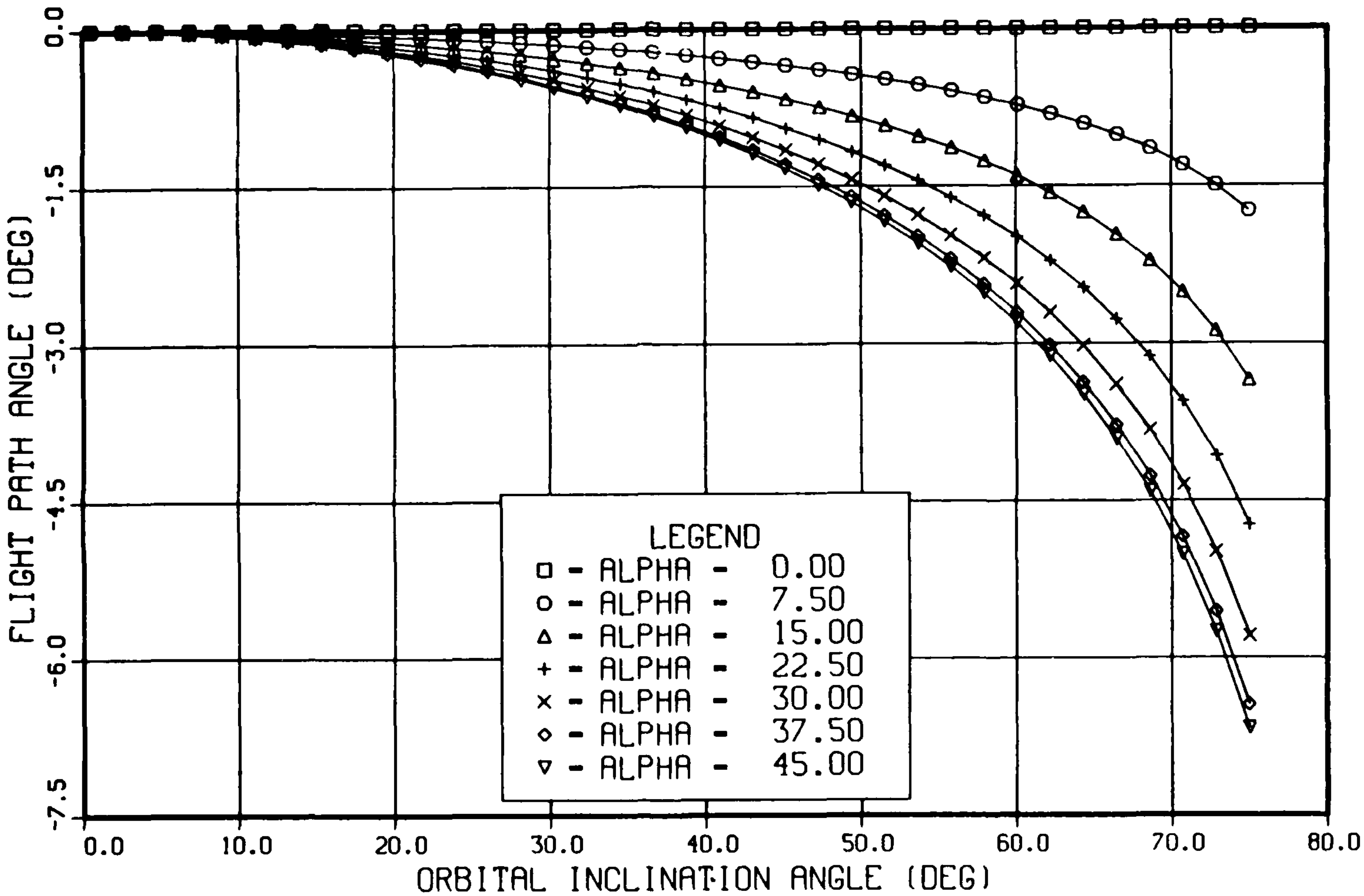

Figure 13. States of Validity for the Non-Rotating Earth, du/dh Equation of Motion ($u = .210$, $h = .0077$)

Effect of h Variation. Holding all other variables constant, the effect of change in non-dimensional altitude on these solutions is negligible. Figures F1 and F2 in Appendix F present results for values of h higher and lower than in Figure 13 for the same values of u and α. Comparison of Figures F1 and F2 indicates that only extremely small changes in flight path angle occur for this large change in h. This ineffectiveness of h variation on γ can also be intuitively seen from examination of Eq (6.3). Note that for all realistic values for non-dimensional altitude, 1.0 is much larger than h in the expression

$$(1 + h)^{3/2}$$

Effect of u Variation. Figures F3 and F4 present solution results for values of speed ratio higher and lower than in Figure 13 for the same range of values of argument of latitude at epoch. Comparison of Figures 13, and Figures F1 - F4 demonstrates that large changes in flight path angle occur for this large range of u. As u decreases, the magnitude of γ increases for curves of constant α .

Effect of α Variation. The trend of the effect of variation in argument of latitude at epoch can be seen by review of Figures 13 and Figures F1 - F4. For the range of values of α in these plots, γ and the change in γ decrease as α increases along lines of constant inclination angle. These trends are more prominent for large values of I. The α terms in Eq (6.2) are

$$- \cos(\alpha)\sin(\alpha)$$

Examination of this term and the properties of the inverse tangent function in Eq (6.2) indicates the solutions given in Figure 13 will be symmetric about $\alpha = 0$ and 180 degrees and these symmetric solutions will repeat about

$$\pm 90 - \alpha \text{ (degrees) for } \pm \alpha$$

Figures 13, F5, F6, and F7 demonstrate these observations.

Trajectory States in Three Dimensions. Figures 14 and 15 present three-dimensional plots of the trajectory states of validity for the du/dh non-rotating Earth equation of motion. Solutions to Eq (6.2) are plotted here for flight path angle versus orbital inclination angle and argument of latitude at epoch. The solution surfaces in these figures are constructed of lines of constant α and lines of constant flight path angle. These contours of constant γ are drawn for changes of γ of 0.165 degrees in Figure 14 and changes of γ of 1.04 degrees in Figure 15. In both of these plots, values of flight path angle are near zero for small inclination angles and for very small values of argument of latitude at epoch. Although Figures 14 and 16 present solutions to Eq (6.2) for vastly different values of speed ratio, the two plots look almost identical; only the scale of flight path angle values differs.

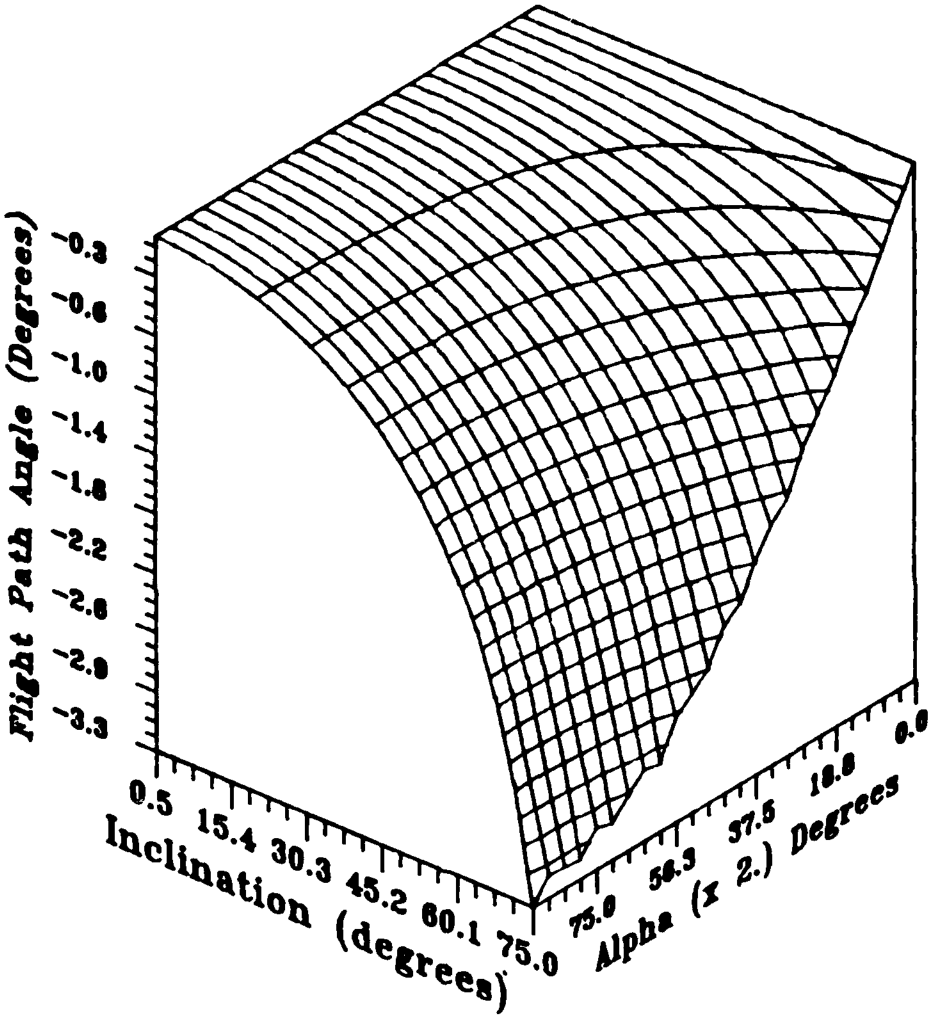

Figure 14. Surface of Trajectory State Solutions of Validity for the Non-Rotating Earth, du/dh Equation of Motion (u = .860 , h = .0129)

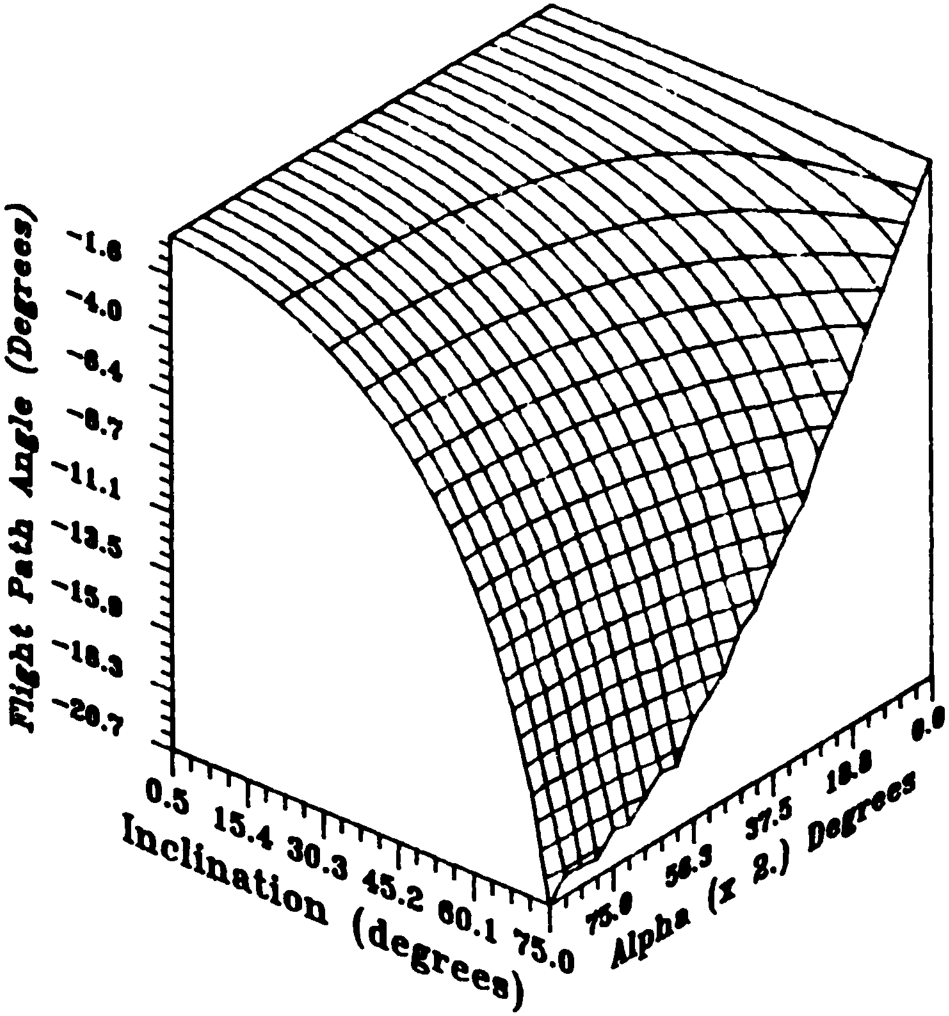

Figure 15. Surface of Trajectory State Solutions of Validity for the Non-Rotating Earth, du/dh Equation of Motion (u = .020 , h = .0052)

Trajectory States of Validity for the dI/dh Non-Rotating Earth Equation of Motion. Trajectory states are given below where the non-rotating Earth dI/dh equation of motion is valid. Trends in these solutions to Eq (6.5) are discussed.

Typical Solution Observations. Figure 16 is a typical plot of solutions of flight path angle versus orbital inclination angle. For constant non-zero values of argument of latitude at epoch, γ increases as I increases. Values of flight path angle decrease in this plot as values of α increase.

Effect of h Variation. Holding all other variables constant, the effect of change in non-dimensional altitude on these solutions is negligable. Figures F8 and F9 present solution results for values of h higher and lower than in Figure 16, and for the same values of u and α. Comparison of Figures F8 and F9 indicates that only extremely small changes in γ occur for this large change in h. As in the case for the du/dh Rotate solutions, this ineffectiveness of h variation on γ can also be intuitively seen from examination of the expression for C in Eq (6.5).

Effect of u Variation. Figures F10 and F11 present solutions for values of speed ratio higher and lower than in Figure 16 for the same range of values of argument of latitude at epoch. Comparison of Figures 16, F10, and F11 indicates that large changes in γ occur for this large range of u. As u decreases (along with the ineffectual non-

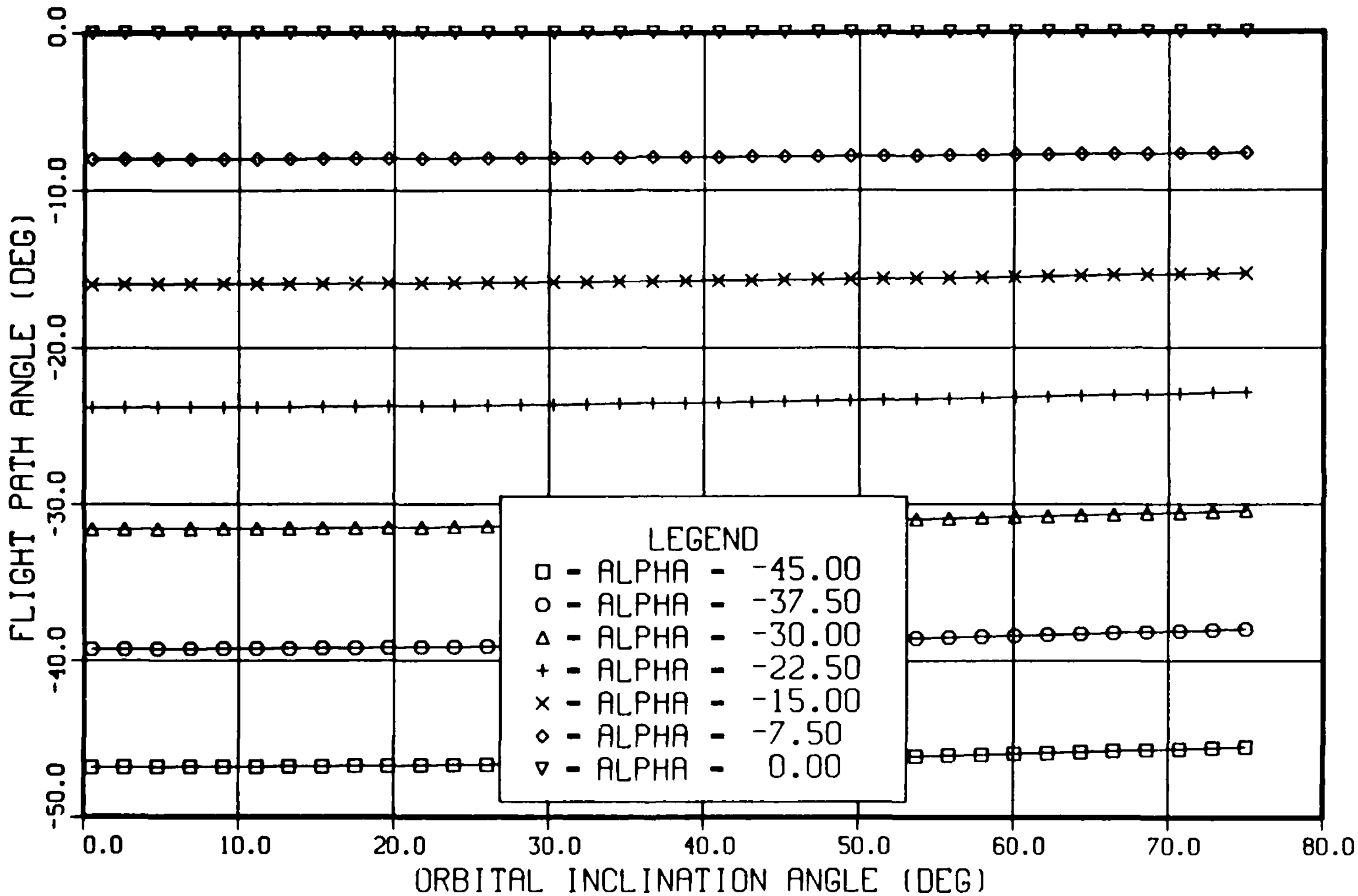

Figure 16. States of Validity for the Non-Rotating Earth, dI/dh Equation of Motion (u = .210 , h = .0077)

dimensional altitude), values of flight path angle increase for curves of constant α.

Effect of α Variation. The trend in the effect of variations of argument of latitude at epoch can be seen by review of Figures 16 and Figures F8 - F11. For the range of α in these plots, γ decreases as α increases for constant I. This change in flight path angle is more prominent for small values of speed ratio. Change in the range of α can also be easily seen analytically. The α terms in Eq (6.5) can be written simply as $\tan(\alpha)$. Examination of this term and the properties of the inverse tangent function in Eq (6.5) indicates the solutions given in Figure 16 will be symmetric about α = 0 and 180 degrees, and that these symmetric solutions will repeat about

$$\alpha \mp 180 \text{ (degrees) for } \pm \alpha$$

Figures 16, F12, F13, and F14 demonstrate these observations.

Trajectory States in Three Dimensions. Figures 17, 18, and 19 present three-dimensional plots of the trajectory states of validity for the dI/dh non-rotating Earth equation of motion. Solutions to Eq (6.5) are plotted here for flight path angle versus orbital inclination angle and for argument of latitude at epoch. The solution surfaces in these figures are constructed of lines of constant flight path angle and lines of constant argument of latitude at epoch. Figures 17, 18, and 19 display plots generated for

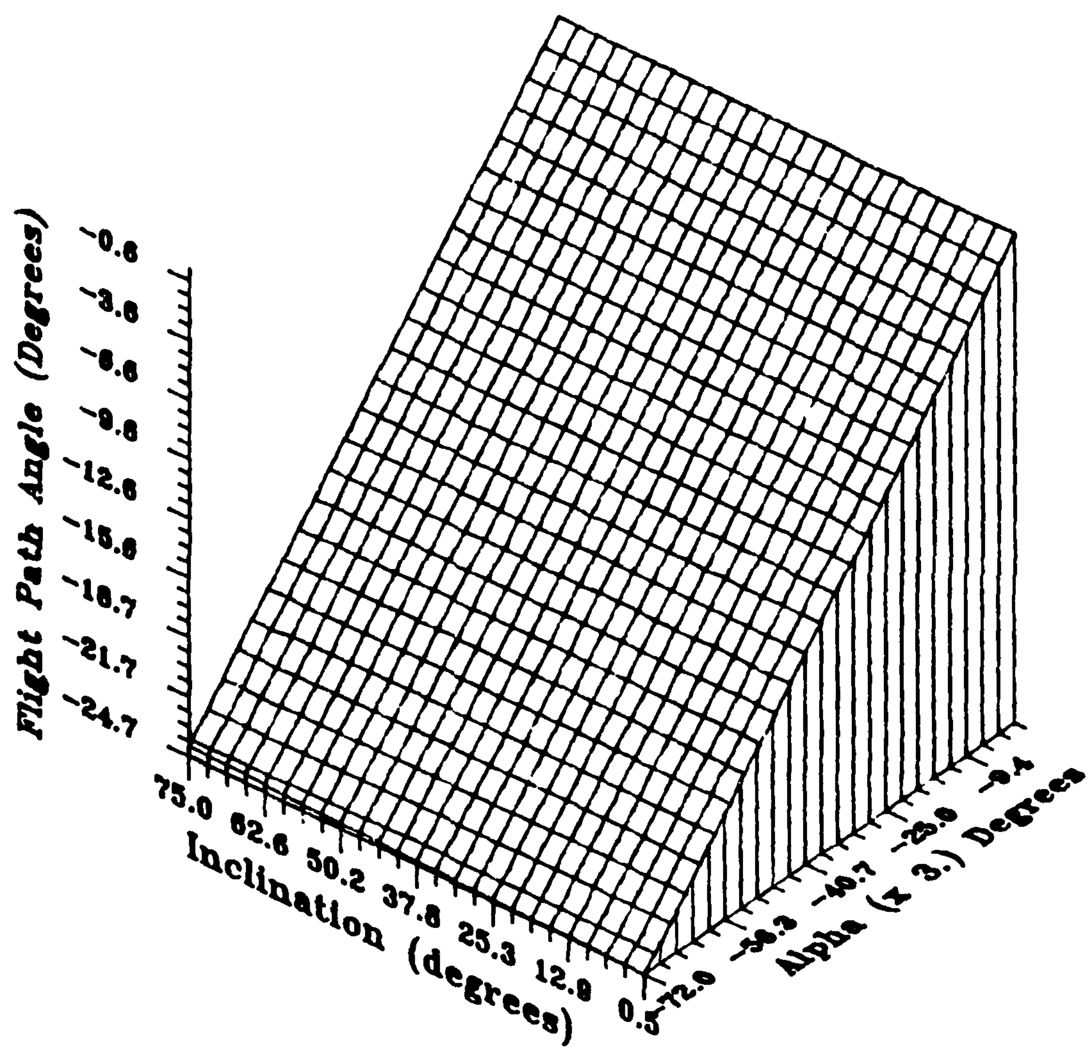

Figure 17. Surface of Trajectory State Solutions of Validity for the Non-Rotating Earth, dI/dh Equation of Motion (u = .860 , h = .0129)

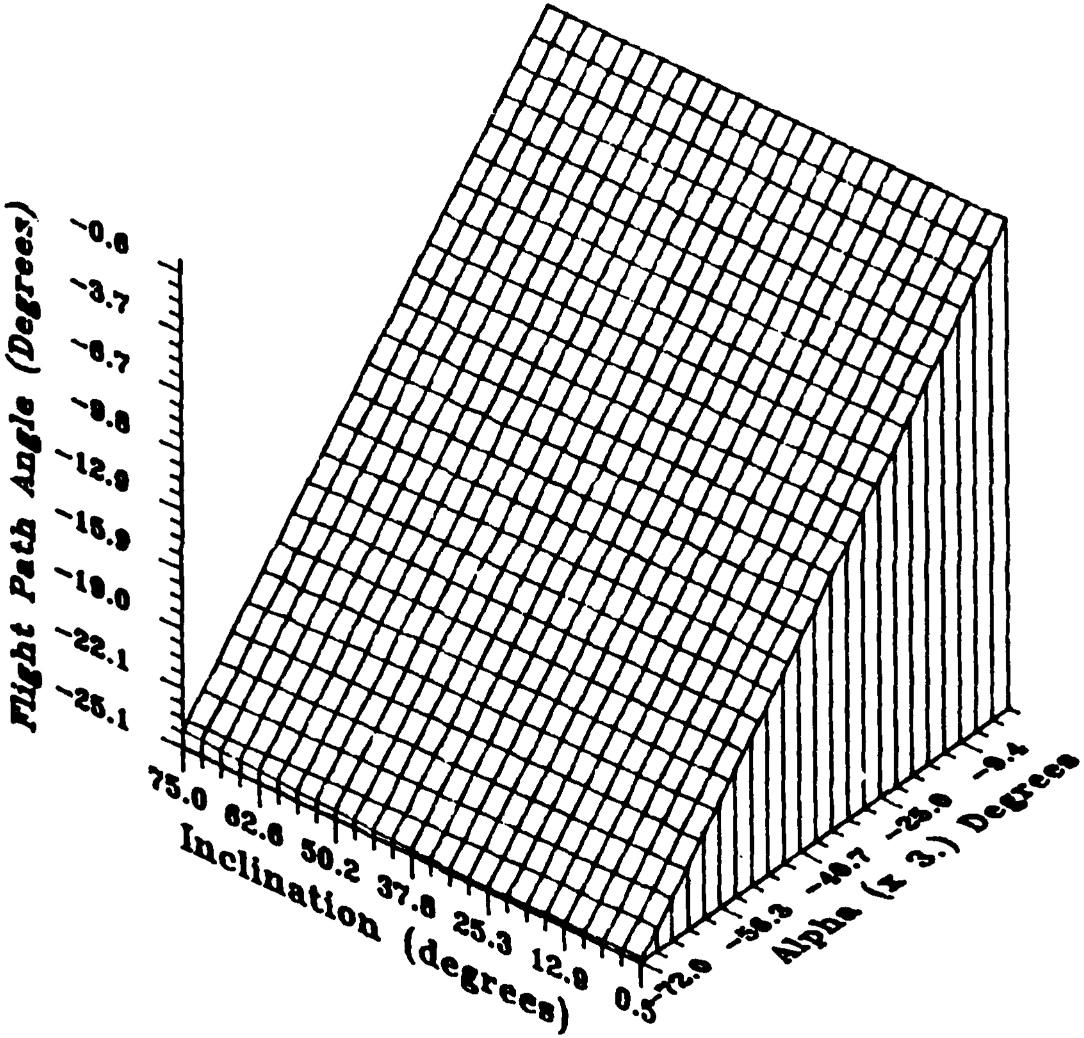

Figure 18. Surface of Trajectory State Solutions of Validity for the Non-Rotating Earth, dI/dh Equation of Motion (u = .300 , h = .0084)

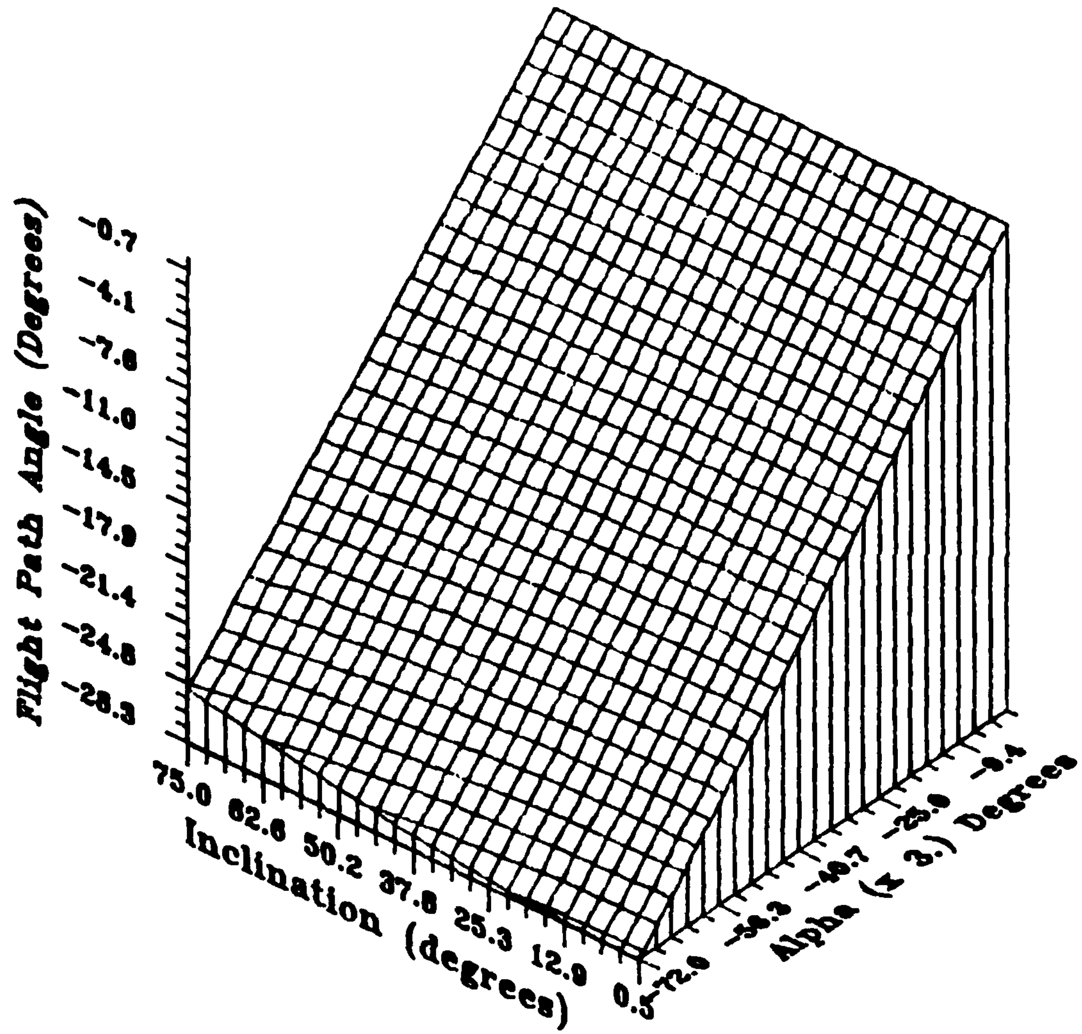

Figure 19. Surface of Trajectory State Solutions of Validity for the Non-Rotating Earth, dI/dh Equation of Motion (u = .020 , h = .0052)

successively decreasing values of speed ratio. Although these values of u differ greatly, the solution surfaces are very similar in appearance.

Solution Estimation Methods

Some effort was spent in developing the computer program to solve Eqs (6.2) and (6.5) for large ranges of values of speed ratio, non-dimensional altitude, orbital inclination angle, flight path angle, and argument of latitude at epoch. To rapidly assess the validity of the non-rotating Earth atmospheric entry equations of motion for a particular trajectory state, there is an advantage to having a quick estimation algorithm. On the following pages, methods are presented to estimate some solutions to the du/dh and dI/dh Rotate expressions given by Eqs (6.2) and (6.5), respectively.

Estimates to the du/dh Rotate Solutions. Figure 20 presents a plot of trajectory state solutions to Eq (6.2), the du/dh Rotate equation. The solution surface in this figure is onstructed of lines of constant flight path angle and lines of constant argument of latitude at epoch. Two separate estimations of solutions to Eq (6.2) can be made with plots such as Figure 20. The first is the approximation that values of flight path angle in the triangular plane (ABC) of trajectory states have some constant, near-zero value. In general, this is a good approximation for γ, especially for high values of u. The

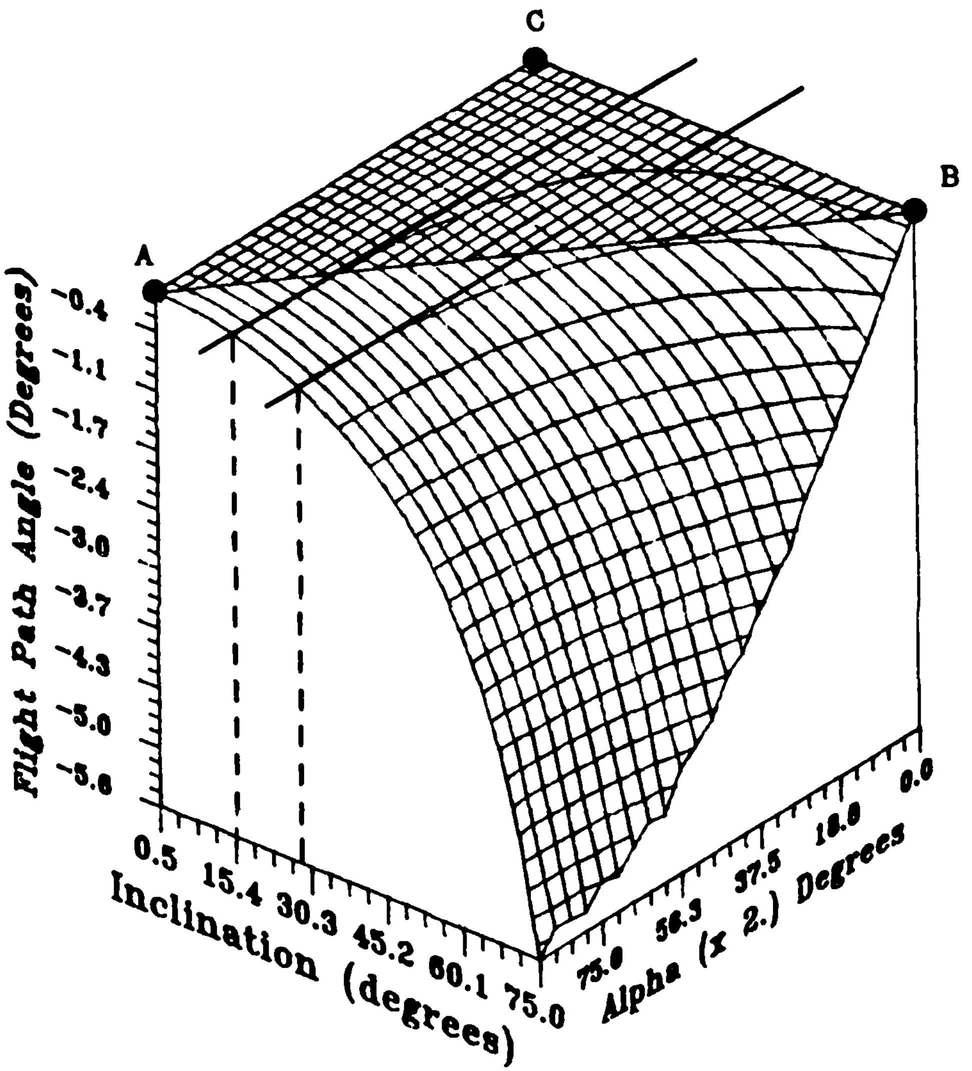

Figure 20. Estimation of Trajectory State Solutions of Validity for the Non-Rotating Earth, du/dh Equation of Motion (u = .300 , h = .0084)

ABC plane of "constant" γ is bounded by

$0.0 \leq \alpha \leq 45.0$ degrees

$0.5 \leq I \leq 75.0$ degrees

Line AB: $\alpha = -(0.60403)I - (45.302)$ degrees

Therefore, the conditions for assuming flight path angle has some near-zero value are given by

$$(.02207)\alpha + (.01333)I \leq 1.00 \tag{6.9}$$

and by the ranges of inclination angle and argument of latitude at epoch given above.

The maximum error in assuming $\gamma = 0.0$ degrees in the ABC plane in Figure 20 is approximately equal to the value of γ corresponding to the second flight path angle contour from the top of the plot. This maximum error in flight path angle is about $\gamma = -0.56$ degrees for this plot. Application of this estimate to the corresponding ABC planes in Figures 14 and 15 give maximum errors in flight path angle of about 0.30 and 1.5 degrees, respectively.

An estimate of a non-zero value for flight path angle in the ABC plane can reduce these maximum errors. Assuming the value of γ in this plane is equal to half the γ value of the second constant γ contour in Figure 20 produces an estimated flight path angle of -0.28 degrees with maximum error of 0.28 degrees. Likewise, estimates of $\gamma = -0.15$ degrees and $\gamma = -0.75$ degrees in the ABC planes in Figures 14 and

15 give maximum errors in flight path angle of about 0.15 degrees and 0.75 degrees, respectively.

Another approximation to the solution of Eq (6.2) is given by the simple assumption that flight path angle has near-zero values for small values of inclination angle. Maximum error in an estimated flight path angle is dependent upon the chosen value of inclination angle for this approximation. For example, assuming $\gamma = 0.0$ degrees for $I \leq 15.4$ degrees invokes a maximum error in flight path angle of about 0.28 degrees in Figure 20. Assuming $\gamma = 0.0$ degrees for $I \leq 29.2$ degrees produces a maximum error in flight path angle of about 0.56 degrees in Figure 20.

An estimate of a non-zero value for flight path angle in the ABC plane can reduce these maximum errors for the same range of inclination angle. Assuming $\gamma = -0.14$ degrees for $I \leq 15.4$ degrees produces a maximum error in γ of about 0.14 degrees in Figure 20. Estimation of a flight path angle value of -0.28 degrees for $I \leq 29.2$ degrees gives a maximum error of 0.28 degrees.

The accuracy of both of these estimations for solutions to the du/dh Rotate equation is heavily dependent on the value of speed ratio for a given trajectory state and the estimate of flight path angle.

<u>Planar Estimation to dI/dh Rotate Solutions</u>. A planar estimation method was developed to obtain rough estimates of solutions to Eq (6.5). For given values of speed ratio, the

following relations give an estimate of the planar surface of solutions that appears in Figures 17 - 19.

$$\gamma = \gamma_0 - \frac{a}{c}(\alpha - \alpha_0) - \frac{b}{c}(I - I_0) \tag{6.10}$$

where

$c = -89.40$ degrees

$I_0 = 52.65$ degrees

$\alpha_0 = -21.00$ degrees

and values of a, b, and c are polynomial functions of u, given by

$$a = (106.701) - (27i.537)u + (1209.57)u^2$$
$$- (1840.30)u^3 + (895.29)u^4$$

$$b = (5.657) - (90.002)u + (401.76)u^2$$
$$- (611.66)u^3 + (297.66)u^4$$

$$\gamma_0 = (-26.104) + (81.784)u - (365.55)u^2$$
$$+ (556.77)u^3 - (271.00)u^4$$

Examples of planar estimates of solutions to Eq (6.5), generated by the above method, are presented in Figures 21 - 23. These estimated solutions correspond to the numerically generated solutions presented in Figures 17 - 19. A more exact method to estimate solutions to Eq (6.5) is given by a three-dimensional curvefit algorithm described below.

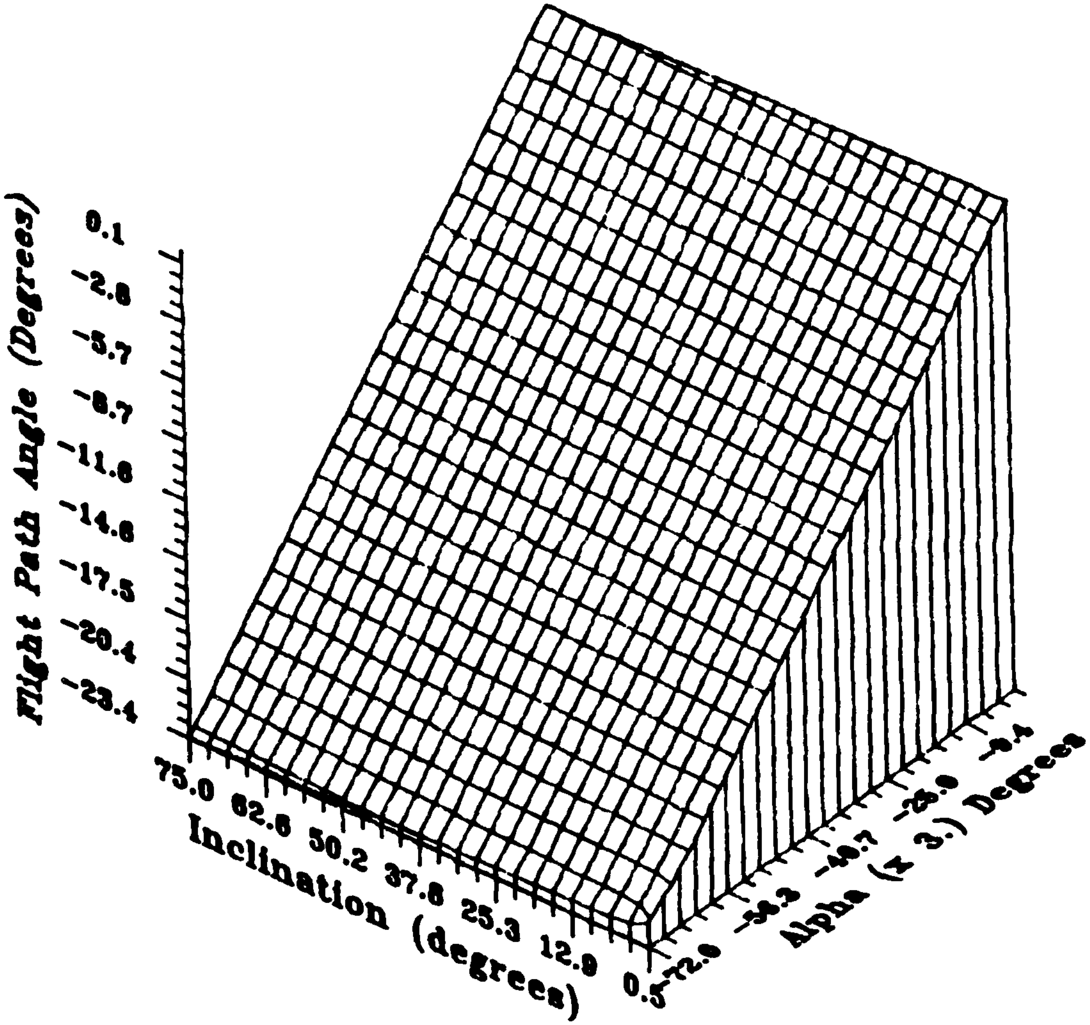

Figure 21. Estimation of Trajectory State Solutions of Validity for the Non-Rotating Earth, dI/dh Equation of Motion (u = .860 , h = .0129)

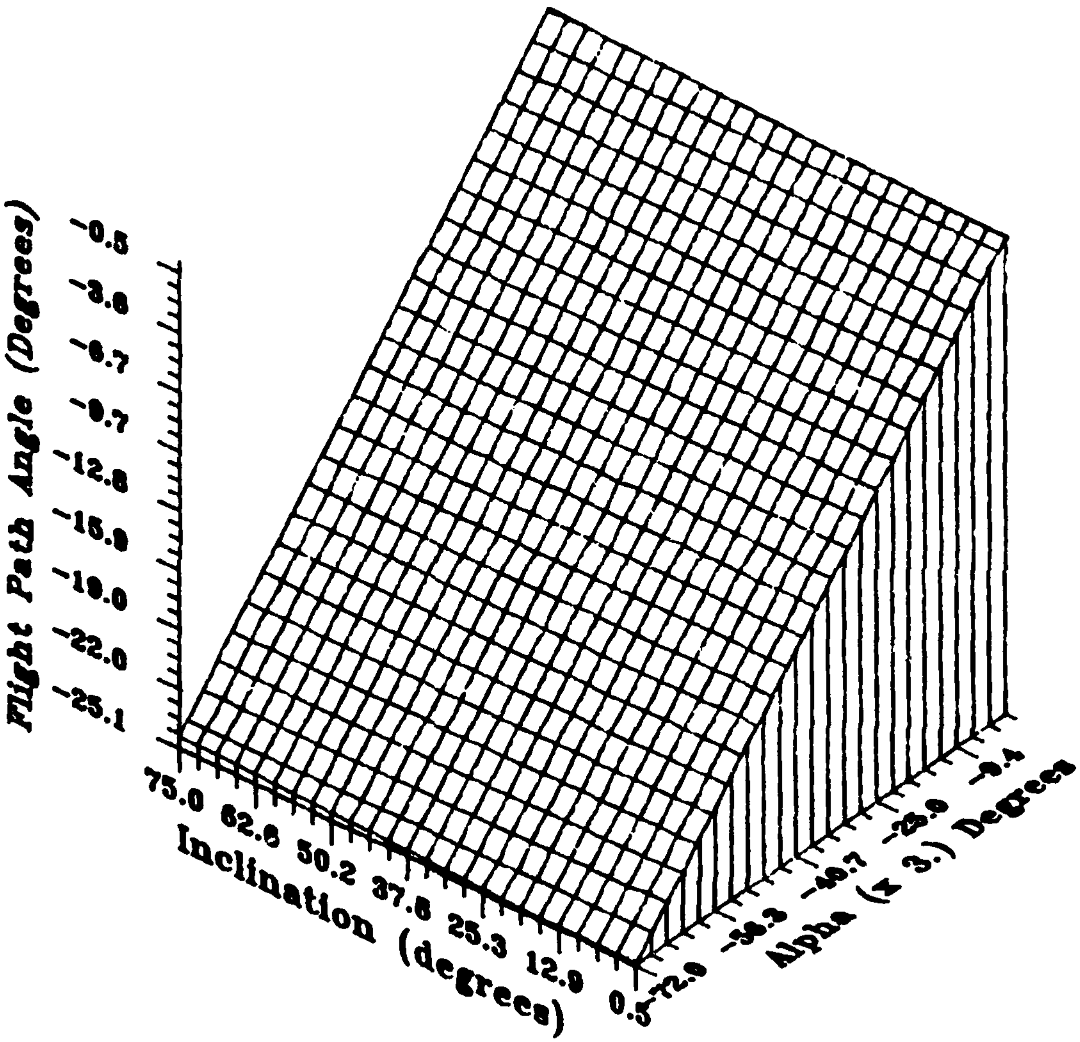

Figure 22. Estimation of Trajectory State Solutions of Validity for the Non-Rotating Earth, dI/dh Equation of Motion (u = .300 , h = .0084)

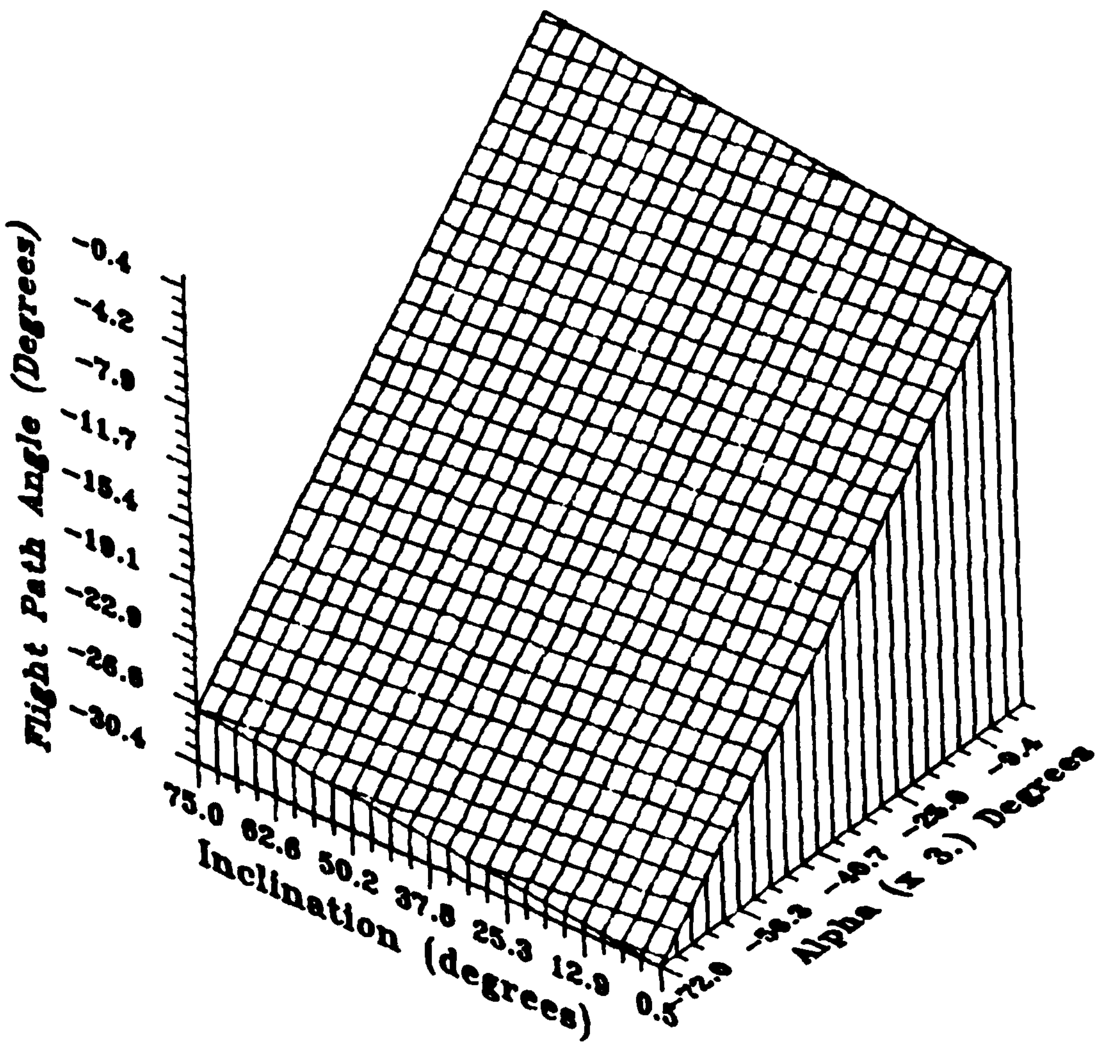

Figure 23. Estimation of Trajectory State Solutions of Validity for the Non-Rotating Earth, dI/dh Equation of Motion $(u = .020 , h = .0052)$

Polynomial Curve Fit Estimation to dI/dh Rotate Solutions. Using polynomial least squares curve fitting routines, a curvefit to the solutions of the dI/dh Rotate equation was developed for ranges of inclination angle and argument of latitude at epoch of $0.5 \leq I \leq 75.0$ degrees and $-45.0 \leq \alpha \leq 0.0$ degrees. This curvefit is three-dimensional in the sense that it predicts solutions for γ from Eq (6.5) for various I, α, and u. Since changes in h were shown to produce negligable changes in γ, h was not included in the fit. The resulting algorithm produces accurate solution estimates. Figure 24 presents numerically generated values of γ versus I, and Figure 25 presents corresponding curvefit generated estimations of γ versus I for the same values of α, u, and h.

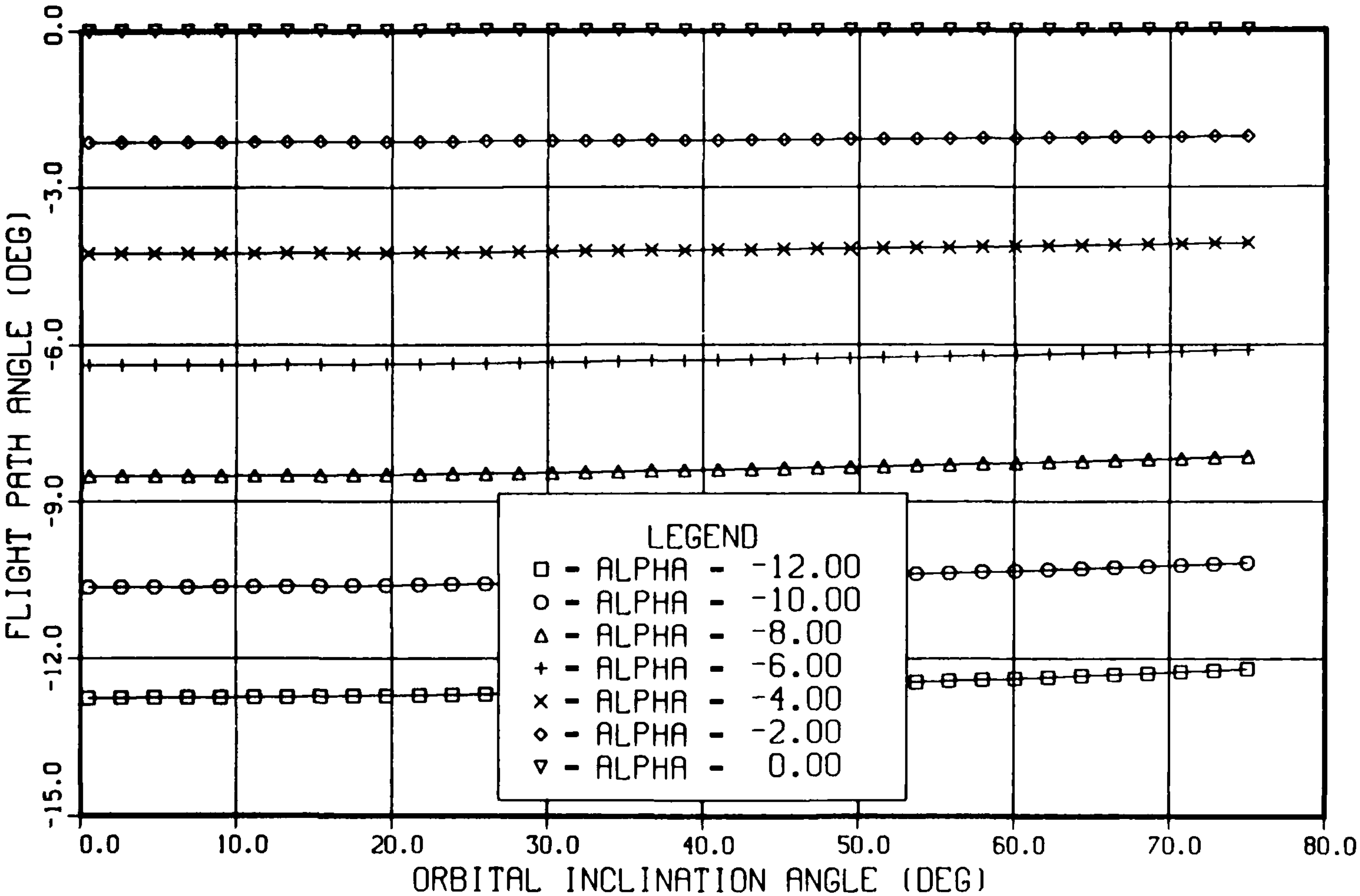

Figure 24. States of Validity for the Non-Rotating Earth, dI/dh Equation of Motion (u = .210 , h = .0077)

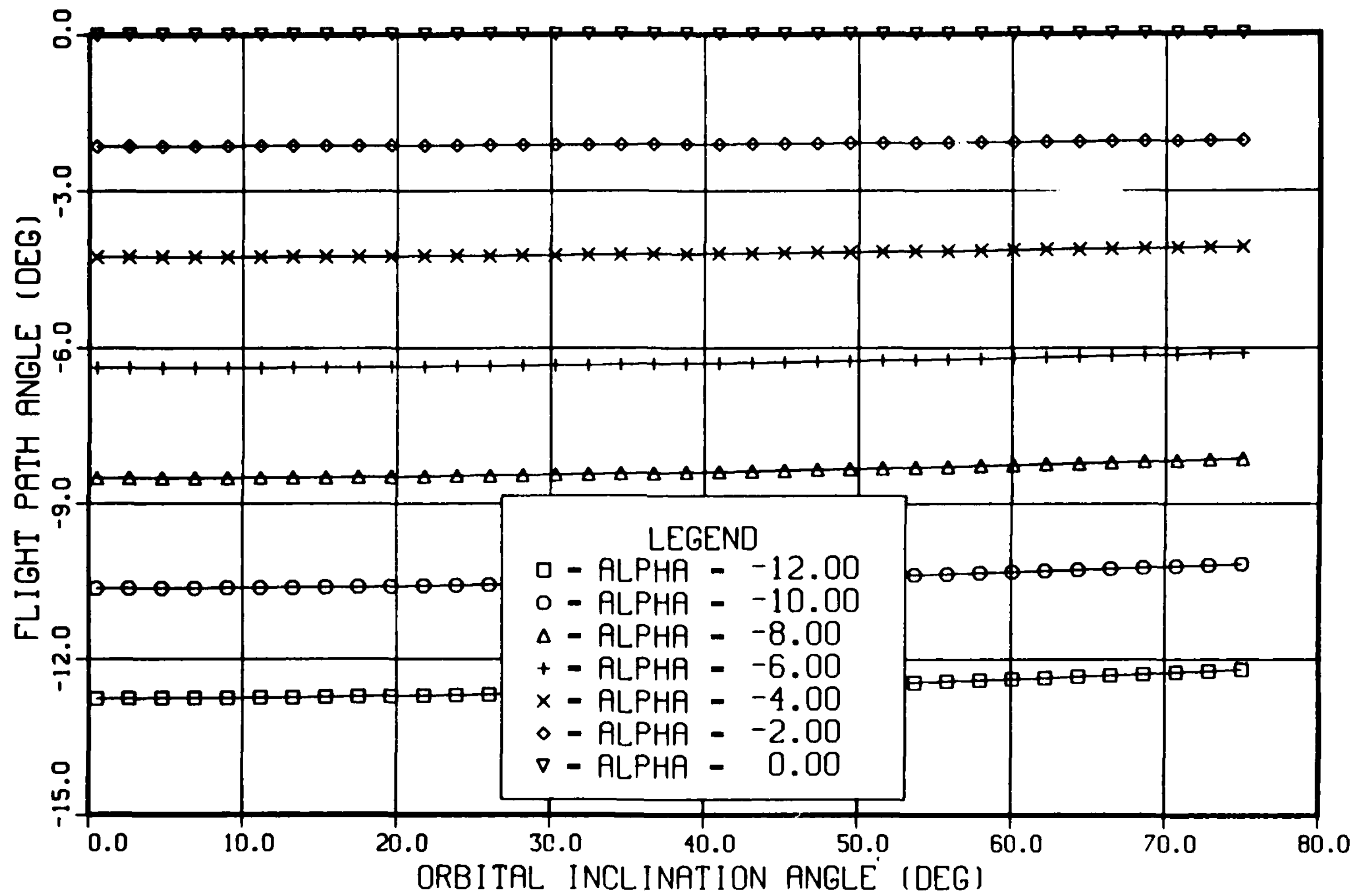

Figure 25. Estimation of Trajectory State Solutions of Validity for the Non-Rotating Earth, dI/dh Equation of Motion ($u = .020$, $h = .0052$)

VII. Conclusions and Recommendations

Conclusions

This effort investigated the validity of the non-rotating planet assumption for three-dimensional Earth atmospheric entry. This study was limited to entry at orbital inclination angles between 0.5 and 75.0 degrees, where most Earth atmospheric entry occurs, and vehicle speeds ranging from circular orbital velocities to low supersonic speeds, where terminal maneuvers (such as landing approaches) are usually initiated. Constant lift-to-drag ratio and ballistic coefficient were assumed along the vehicle's flight path. Validity results are coordinate dependent since singularities exist in the equations of motion. On the basis of this investigation, the following conclusions are made:

1. As a set, the five non-rotating planet equations of motion (Section IV) are invalid for Earth atmospheric entry. Hence, the non-rotating Earth assumption, common in analytical entry analyses, produces incorrect entry trajectory results. Solutions to the non-rotating Earth equations of motion were derived from expansions of the rotating Earth equations of motion using the method of matched asymptotic expansions. These solutions are given by Eqs (5.86) - (5.90).

2. The dq/dh equation of motion for a non-rotating Earth is never valid for the ranges of inclination angle and vehicle speeds mentioned above. Hence, a solution, Eq (5.95), was developed for the dq/dh equation of motion for three-dimensional lifting atmospheric entry of a rotating spherical Earth. This solution is valid for the entire non-dimensional altitude domain and it accounts for the Coriolis acceleration on the flight vehicle. The method of matched asymptotic expansions was employed in this derivation.

3. A variety of realistic Earth atmospheric entry trajectory states exist where some of the non-rotating equations of motion are valid for a rotating Earth. The $d\Omega/dh$, $d\alpha/dh$, and dI/dh non-rotating equations of motion are valid for the same entry trajectory states for a rotating Earth. Other trajectory states exist where the du/dh non-rotating Earth equation of motion is valid for a rotating Earth. These two sets of trajectory states do not overlap for the investigated ranges of entry inclination angle and velocity.

4. A number of trajectory states are presented where the du/dh non-rotating equation of motion and the $d\Omega/dh$, $d\alpha/dh$, and dI/dh non-rotating equations of motion are valid. Holding speed ratio constant, realistic

variations of non-dimensional altitude causes negligible changes in these states of validity. The magnitudes of the flight path angles where these non-rotating Earth equations of motion are valid increase as speed ratio values decrease. Trajectory states of validity are symmetrical about various values of argument of latitude at epoch for the non-rotating Earth dI/dh and du/dh equations of motion. Two methods to estimate flight path angle values where these equations of motion are valid are also presented.

Recommendations

Based on the observations of the assumptions and of the findings of this investigation, the following recommendations for further study are proposed:

1. Investigation of the validity of the non-rotating planet assumption for three-dimensional Earth atmospheric entry was limited to orbital inclination angles between 0.5 and 75.0 degrees. Further investigation should be undertaken for other ranges of inclination. Near-polar and negative inclination angles would be of particular interest.

2. Further study of the trajectory states where the du/dh and the $d\Omega/dh$, $d\alpha/dh$, and dI/dh non-rotating Earth equations of motion are valid is warranted. An

attempt should be made to construct entire trajectories from these trajectory states. If successful, a reduction in the complexity of the equations of motion could be made for certain trajectories and possibly even specific trajectory classes. This reduction would allow the non-rotating planet equations of motion to be used in place of the more complex rotating planet equations of motion, causing past analytical studies that utilized the non-rotating planet assumption to be valid for these trajectories. Current trajectory optimization computer programs consume many valuable hours of computer run-time, iterativly integrating the rotating planet equations of motion. Optimization computer programs such as the Air Force Flight Dynamics Laboratory's ENTRAN (ENtry TRajectory ANalysis) and OTIS (Optimal Trajectories via Implicit Simulation) codes, could greatly benefit from a reduction in the complexity of the rotating planet equations of motion.

3. The method of matched asymptotic expansions has proven to be a powerful tool in the development of solutions to boundary layer problems such as planetary atmospheric entry. Further application of this method should be undertaken to obtain a full set

of solutions to the equations of motion for three-dimensional lifting atmospheric entry for a rotating Earth. This activity would entail finding solutions to the complex differential equations given by Eqs (5.20) - (5.24) and Eqs (5.46) - (5.52), and then matching and forming composite solutions from these results.

Appendix A

Derivation of Equations Relating θ, ϕ, ψ and α, Ω, I Using Spherical Trigonometric Relationships

This appendix presents the basic derivation of equations relating θ, ϕ, ψ and α, Ω, I using trigonometric relationships. In this paper, the planetary model is spherical and hence the spherical sine and cosine laws can be successfully applied.

Figure A1 depicts a reference spherical triangle. For clarity, the angles between the curved line segments are called interior angles whereas the angles formed between the unit vectors, $\hat{r}_i$, $\hat{r}_j$, and $\hat{r}_k$, are termed exterior angles. Hence, A, B, and C are exterior angles and a, b, and c are interior angles in Figure A1.

The sine law for spherical triangles states that the ratio of the sine of an interior angle to the sine of its opposing exterior angle is constant.

$$\frac{\sin(a)}{\sin(A)} = \frac{\sin(b)}{\sin(B)} = \frac{\sin(c)}{\sin(C)} \tag{A.1}$$

The cosine law for a spherical triangle states that the cosine of an interior angle is the sum of two products. The first product is formed of the cosines of the other two interior angles; the second product is formed of the sines of the other two interior angles and the cosine of the

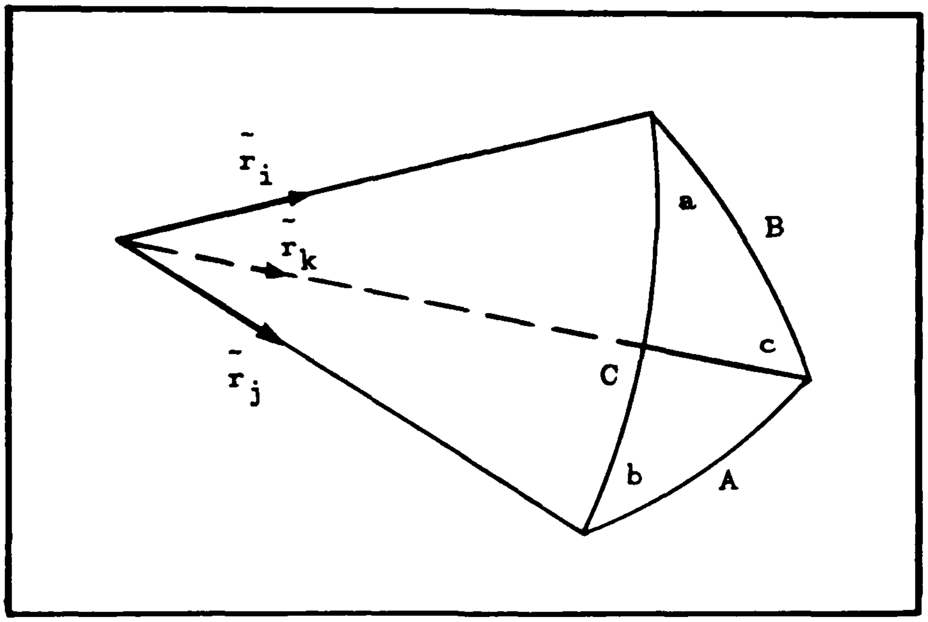

Figure A1. Reference Spherical Triangle

opposing exterior angle. The cosine law is given by the following expression.

$$\cos(a) = \cos(b)\cos(c) + \sin(b)\sin(c)\cos(A) \qquad (A.2)$$

Eqs (A.1) and (A.2) are used to relate the variables θ, ϕ, ψ, α, Ω, and I. From comparison of Figures 11 and 12 (Section II) to Figure A1, the exterior and interior angles of the formed spherical triangle are

Exterior Angles	Interior Angles
$A = \theta - \Omega$	$a = \pi/2 - \psi$
$B = \phi$	$b = I$
$C = \alpha$	$c = \pi/2$

where

$\theta \equiv$ longitude

$\phi \equiv$ latitude

$\psi \equiv$ heading angle

$\alpha \equiv$ argument of latitude at epoch

$\Omega \equiv$ longitude of the ascending node

$I \equiv$ inclination angle

Application of the sine law to these variables gives the following:

$$\frac{\sin\phi}{\sin I} = \frac{\sin\alpha}{\sin(\pi/2)} = \frac{\sin(\theta - \Omega)}{\sin(\pi/2 - \psi)}$$

Since

$$\cos\psi = \sin(\tfrac{\pi}{2} - \psi) \quad \text{and} \quad \sin(\tfrac{\pi}{2}) = 1$$

$$\sin\alpha = \frac{\sin(\theta - \Omega)}{\cos\psi} \tag{A.3}$$

$$\sin\phi = \sin I \sin\alpha \tag{A.4}$$

Application of the cosine law to the inclination angle gives the following.

$$\cos(I) = -\cos(\tfrac{\pi}{2})\cos(\tfrac{\pi}{2} - \psi) + \sin(\tfrac{\pi}{2})\sin(\tfrac{\pi}{2} - \psi)\cos(\phi)$$

Since $\sin\psi = \cos(\frac{\pi}{2} - \psi)$ and $\cos(\frac{\pi}{2}) = 0$

$$\cos(I) = \cos(\psi)\cos(\phi) \tag{A.5}$$

Another application of the cosine law gives

$$\cos(\tfrac{\pi}{2}) = -\cos(I)\cos(\tfrac{\pi}{2} - \psi) + \sin(I)\sin(\tfrac{\pi}{2} - \psi)\cos(\alpha)$$

which reduces to $\cos(I)\sin(\psi) = \sin(I)\cos(\psi)\cos(\alpha)$

or $$\cos\alpha = \frac{\tan\psi}{\tan I} \tag{A.6}$$

Yet another application of the cosine law gives

$$\cos(\frac{\pi}{2} - \psi) = -\cos(\frac{\pi}{2})\cos(I) + \sin(\frac{\pi}{2})\sin(I)\cos(\theta - \Omega)$$

$$\sin(\psi) = \sin(I)\cos(\theta - \Omega) \tag{A.7}$$

Some other relationships can be found that are coupled to Eqs (A.3) - (A.7) and are useful for some occasions. Substituting Eq (A.5) into Eq (A.6)

$$\cos\alpha = \cos\phi\cos(\theta - \Omega) \tag{A.8}$$

Substituting Eq (A.5) into Eq (A.3) gives

$$\sin(\theta - \Omega) = \frac{\tan\phi}{\tan I} \tag{A.9}$$

Rewriting Eq (A.6) and substituting in Eq (A.5) gives the expression

$$\sin(\psi) = \frac{\cos\alpha\sin I}{\cos\phi}$$

Substituting Eq (A.4) into this result

$$\sin(\psi) = \frac{\tan\phi}{\tan\alpha} \tag{A.10}$$

The relations given by Eqs (A.3) - (A.10) are used in Section III to transform the equations of motion from terms of latitude, longitude, and heading angle to terms containing orbital inclination angle, longitude of the ascending node, and argument of latitude at epoch.

Appendix B

Some Common Expansions

This appendix presents the Taylors series expansion formula and some basic expansions of common functions that appeared in the equations of motion.

The Taylor series expansion of $f(x)$ about $x = x_0$ is given by the following expression:

$$f(x) = \sum_{n=0}^{\infty} f^{(n)}(x_0) \cdot \frac{(x - x_0)^n}{n!}$$

where $f^{(n)}(x_0) = \dfrac{d^n f(x_0)}{dx^n}$ and $f^0(x_0) = f(x_0)$

Application of this equation to various functions within the equations of motion was facilitated with the following two trigonometric identities:

$$\cos(a + \epsilon b) = \cos(a)\cos(\epsilon b) - \sin(a)\sin(\epsilon b)$$

$$\sin(a + \epsilon b) = \sin(a)\cos(\epsilon b) + \cos(a)\sin(\epsilon b)$$

The Taylor series expansion of the cosine and sine functions gives the following familiar expressions:

$$\cos(\epsilon b) = \frac{\epsilon^0 b^0}{0!} - \frac{\epsilon^2 b^2}{2!} + \frac{\epsilon^4 b^4}{4!} - \frac{\epsilon^6 b^6}{6!} + \ldots$$

Hence

$$\cos(\epsilon b) = 1 + O(\epsilon^2)$$

$$\sin(\epsilon b) = \frac{\epsilon^1 b^1}{1!} - \frac{\epsilon^3 b^3}{3!} + \frac{\epsilon^5 b^5}{5!} - \frac{\epsilon^7 b^7}{7!} + \ldots$$

Therefore

$$\sin(\epsilon b) = \epsilon b + O(\epsilon^3)$$

As an example, the following expansion of the cosine function is derived to $O(\epsilon^2)$.

$$\cos\left[a + \epsilon b + \epsilon^2 c\right] = \cos(a)\cos\left[\epsilon b + \epsilon^2 c\right] - \sin(a)\sin\left[\epsilon b + \epsilon^2 c\right]$$

$$= \cos(a)\left[\cos(\epsilon b)\cos(\epsilon^2 c) - \sin(\epsilon b)\sin(\epsilon^2 c)\right]$$

$$- \sin(a)\left[\sin(\epsilon b)\cos(\epsilon^2 c) + \cos(\epsilon b)\sin(\epsilon^2 c)\right]$$

$$= \cos(a) - \epsilon b \cdot \sin(a) + O(\epsilon^2)$$

The following relations were obtained through application of the Taylor series expansion:

$$\cos\left[a + \epsilon b + O(\epsilon^2)\right] = \cos(a) - \epsilon b \cdot \sin(a) + O(\epsilon^2) \tag{B.1}$$

$$\sin\left[a + \epsilon b + O(\epsilon^2)\right] = \sin(a) + \epsilon b \cdot \cos(a) + O(\epsilon^2) \tag{B.2}$$

$$\tan\left[a + \epsilon b + O(\epsilon^2)\right] = \tan(a) + \frac{\epsilon b}{\cos^2(a)} + O(\epsilon^2) \tag{B.3}$$

$$\left[\tan\left[a + \epsilon b + O(\epsilon^2)\right]\right]^{-1} = \frac{1}{\tan(a)} - \frac{\epsilon b}{\sin^2(a)} + O(\epsilon^2) \tag{B.4}$$

$$\left[\sin\left[a + \epsilon b + O(\epsilon^2)\right]\right]^{-1} = \frac{1}{\sin(a)} - \epsilon b \cdot \frac{\cos(a)}{\sin^2(a)} + O(\epsilon^2) \tag{B.5}$$

$$\left[\cos\left[a + \epsilon b + O(\epsilon^2)\right]\right]^{-1} = \frac{1}{\cos(a)} - \epsilon b \cdot \frac{\sin(a)}{\cos^2(a)} + O(\epsilon^2) \tag{B.6}$$

$$\left[a + \epsilon b + O(\epsilon^2)\right]^{-1} = \frac{1}{a} - \frac{\epsilon b}{a^2} + O(\epsilon^2) \tag{B.7}$$

$$\left[a + \epsilon b + O(\epsilon^2)\right]^{-1/2} = a^{-1/2} - \frac{\epsilon b}{2} \cdot a^{-3/2} + O(\epsilon^2) \tag{B.8}$$

Appendix C

Derivation of Solutions to Selected Outer Expansion Differential Equations

Outer Expansion ϵ^0 Terms

The ϵ^0 terms of the outer expansions found in Section V are as follows:

$$\frac{du_0}{dh} = \frac{-u_0}{(1+h)} \tag{C.1}$$

$$\frac{dq_0}{dh} = \frac{q_0}{(1+h)}\left[\frac{q_0^{\;2}}{u_0} - 1\right] \tag{C.2}$$

$$\frac{dI_0}{dh} = 0 \tag{C.3}$$

$$\frac{d\Omega_0}{dh} = 0 \tag{C.4}$$

$$\frac{d\alpha_0}{dh} = \frac{1}{(1+h)\tan\gamma_0} \tag{C.5}$$

Solutions to this set of differential equations are derived below.

du/dh Equation Solution. Rearranging Eq (C.1) gives

$$\frac{du_0}{u_0} = \frac{-dh}{(1+h)} \qquad \text{or} \qquad \ln(u_0) = -\ln(1+h) + K$$

Solution:

$$u_0(1+h) = C_1 \tag{C.6}$$

dq/dh Equation Solution. From Eq (C.2)

$$\frac{dq_0}{dh} = \frac{q_0}{(1+h)}\left[\frac{q_0^{\ 2}}{u_0} - 1\right]$$

Application of Bernoulli's equation to this differential equation makes the solution derivation straightforward. Bernoulli's equation is given by

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

This form gives the solution (Beyer, 1984:315)

$$v\exp\left[(1-n)\int P(x)dx\right] = (1-n)\int Q(x)\left[(1-n)\int P(x)dx\right]dx + C$$

where $v = y^{1-n}$ and $n \neq 1$

Hence, $\frac{dq_0}{dh} + P(h)q_0 = Q(h)q_0^{\ n}$

where $n = 3$, $v = q_0^{\ -2}$,

$P(h) = \frac{1}{(1+h)}$, and $Q(h) = \frac{1}{C_1} = \frac{1}{u_0\,(1+h)}$

Solution: $\int\frac{dh}{(1+h)} = \ln(1+h)$

$$q_0^{-2}\exp\left[-2\ln(1+h)\right] = \frac{-2}{C_1}\int\exp\left[-2\ln(1+h)\right]dh + K$$

$$\left[q_0\,(1+h)\right]^{-2} = -\frac{2}{C_1}\int(1+h)^{-2}\,dh + K$$

Integrating gives

$$\frac{1}{q_0^{\,2}} = \frac{2}{C_1}(1+h) - C_2(1+h)^2 \tag{C.7}$$

or $$q_0 = \left[\frac{2}{C_1}(1+h) - C_2(1+h)^2\right]^{-1/2} \tag{C.8}$$

<u>dI/dh Equation Solution</u>. From Eq (C.3)

$$\frac{dI_0}{dh} = 0 \quad ; \quad I_0 = C_5 \tag{C.9}$$

<u>dΩ/dh Equation Solution</u>. From Eq (C.4)

$$\frac{d\Omega_0}{dh} = 0 \quad ; \quad \Omega_0 = C_4 \tag{C.10}$$

<u>dα/dh Equation Solution</u>. From Eq (C.5)

$$\frac{d\alpha_0}{dh} = \frac{1}{(1+h)\tan\gamma_0}$$

By definition, $q = \cos\gamma$. The expansions of q and $\cos\gamma$ are given by the following expressions:

$$q_0 + \epsilon q_1 + O(\epsilon^2) = \cos\left[\gamma_0 + \epsilon\gamma_1 + O(\epsilon^2)\right]$$

$$= \cos\gamma_0 - \epsilon\gamma_1\sin\gamma_0 + O(\epsilon^2)$$

By noting the order of ϵ in this equation, it can be seen that

$$q_0 = \cos\gamma_0 \tag{C.11}$$

and $$q_1 = -\gamma_1\sin\gamma_0 \tag{C.12}$$

From Eqs (C.11) and (C.12), an expression is found for the tangent of γ_0.

$$\tan\gamma_0 = \frac{(1 - \cos^2\gamma_0)^{1/2}}{\cos\gamma_0} = \frac{(1 - q_0{}^2)^{1/2}}{q_0}$$

$$\tan\gamma_0 = \left[\frac{1}{q_0{}^2} - 1\right]^{1/2} \tag{C.13}$$

where q_0 was previously found to be

$$q_0 = \left[\frac{2}{C_1}(1+h) - C_2(1+h)^2\right]^{-1/2} \tag{C.14}$$

Therefore

$$\int d\alpha_0 = \int\left[\frac{2}{C_1}(1+h) - C_2(1+h)^2 - 1\right]^{-1/2} \cdot \frac{dh}{(1+h)}$$

Making $x = (1+h)$ causes $dx = dh$ and allows this integral to be rewritten as

$$\alpha_0 = \int\left[\frac{2}{C_1}\cdot x - C_2 x^2 - 1\right]^{-1/2} \cdot \frac{dx}{x}$$

The integral is now in the form of

$$\alpha_0 = \int\frac{dx}{x(X)^{1/2}} \tag{C.15}$$

where $X = a + bx + cx^2$, $f = 4ac - b^2$, $k = 4c/f$,

$a = -1$, $b = 2/C_1$, and $c = -C_2$,

$$f = \frac{4}{C_1^{\,2}}(C_1^{\,2}C_2 - 1), \quad \text{and} \quad k = -C_1^{\,2}C_2/(C_1^{\,2}C_2 - 1)$$

The solution to Eq (C.15) is

$$\alpha_0 = \int \frac{dh}{x(X)^{1/2}} = -\ (-a)^{-1/2}\cos^{-1}\left[(bx + 2a)/\{|x|(-q)^{1/2}\}\right] + C_3$$

Hence

$$\alpha_0 = -\ \cos^{-1}\left[\frac{2x/C_1 - 2}{2|x|(C_2 - 1/C_1^{\,2})^{1/2}}\right] + C_3$$

This solution can be expressed in either of the following two forms:

$$\alpha_0 = -\ \cos^{-1}\left[\frac{1 - C_1/(1+h)}{(1 - C_1^{\,2}C_2)^{1/2}}\right] + C_3 \qquad \text{(C.16)}$$

$$\text{or} \qquad u_0 = 1 + \cos(\alpha_0 - C_3)\cdot\left[1 - C_1^{\,2}C_2\right]^{1/2} \qquad \text{(C.17)}$$

Outer Expansion ϵ^1 Terms

The ϵ^1 terms of the outer expansions found in Section IV are as follows:

$$\frac{du_1}{dh} = \frac{-u_1}{(1+h)} - 4\left[3u_0(1+h)\right]^{1/2}\cdot\cos I_0 \qquad \text{(C.18)}$$

$$\frac{dI_1}{dh} = 2\left[\frac{3(1+h)}{u_0}\right]^{1/2}\frac{\cos\alpha_0 \sin I_0}{\tan\gamma_0}(\cos\alpha_0\tan\gamma_0 - \sin\alpha_0) \qquad \text{(C.19)}$$

$$\frac{d\alpha_1}{dh} = \frac{-\gamma_1}{(1+h)\sin^2\gamma_0} - \frac{\tan\alpha_0}{\tan I_0}\cdot\frac{dI_1}{dh} \tag{C.20}$$

or $$\frac{d\alpha_1}{dh} = \frac{-\gamma_1}{(1+h)\sin^2\gamma_0}$$

$$- 2\left[\frac{3(1+h)}{u_0}\right]^{1/2}\cdot\frac{\sin\alpha_0\cos I_0}{\tan\gamma_0}(\cos\alpha_0\tan\gamma_0 - \sin\alpha_0)$$

$$\frac{d\Omega_1}{dh} = \frac{\tan\alpha_0}{\sin I_0}\cdot\frac{dI_1}{dh} \tag{C.21}$$

or $$\frac{d\Omega_1}{dh} = 2\left[\frac{3(1+h)}{u_0}\right]^{1/2}\frac{\sin\alpha_0}{\tan\gamma_0}(\cos\alpha_0\tan\gamma_0 - \sin\alpha_0)$$

$$\frac{dq_1}{dh} = \frac{q_0}{(1+h)}\left[\frac{-q_0{}^2 u_1}{u_0{}^2} + \frac{2q_1}{u_0}\right] + \frac{q_1}{(1+h)}\left[\frac{q_0{}^2}{u_0} - 1\right]$$

$$- 2q_0\cos I_0\left[\frac{3(1+h)}{u_0}\right]^{1/2} \tag{C.22}$$

In order to find the solution to the dq_1/dh equation, a solution to Eq (C.18), the du_1/dh equation, must first be found.

du/dh Equation Solution.

$$\frac{du_1}{dh} = \frac{-u_1}{(1+h)} - 4\left[3u_0(1+h)\right]^{1/2}\cdot\cos I_0$$

Since $u_0(1+h) = C_1$ and $I_0 = C_5$ we find that

$$\frac{du_1}{dh} + \frac{u_1}{(1+h)} = -4\cos(C_5)\cdot(3C_1)^{1/2}$$

This differential equation is in the form of

$$\frac{dy}{dx} + P(x)y = Q(x)$$

This form gives the solution

$$y\cdot\exp\left[\int P(x)\,dx\right] = \int Q(x)\left[\int P(x)\,dx\right]dx + C$$

For the du_1/dh differential equation

$$y = u_1, \quad x = h,$$

$$P(h) = \frac{1}{(1+h)}, \quad \text{and} \quad Q = -4\cos(C_5)\cdot(3C_1)^{1/2}$$

$$\exp\left[\int\frac{dh}{(1+h)}\right] = \exp\left[\ln(1+h)\right] = (1+h)$$

$$(1+h)u_1 = Q\int(1+h)\,dh = \frac{Q}{2}(1+h)^2 + C_6$$

Solution:

$$u_1 = \frac{C_6}{(1+h)} - 2(3C_1)^{1/2}\cos(C_5)(1+h) \tag{C.23}$$

<u>dq/dh Equation Solution</u>.

$$\frac{dq_1}{dh} = \frac{q_0}{(1+h)}\left[-\frac{q_0{}^2u_1}{u_0{}^2} + \frac{2q_1}{u_0}\right] + \frac{q_1}{(1+h)}\left[\frac{q_0{}^2}{u_0} - 1\right]$$

$$-\ 2q_0\cos I_0\left[\frac{3(1+h)}{u_0}\right]^{1/2} \tag{C.22}$$

where u_0, u_1, and I_0 are given by the following equations:

$$u_0 = C_1/(1+h) \tag{C.6}$$

$$u_1 = \frac{C_6}{(1+h)} - 2(3C_1)^{1/2}\cos(C_5)(1+h) \tag{C.23}$$

$$I_0 = C_5 \tag{C.9}$$

Substitution of Eqs (C.6), (C.9), and (C.23), into Eq (C.22) gives

$$\frac{dq_1}{dh} = 2\frac{q_1 q_0}{C_1} - \frac{q_0{}^3 C_6}{C_1{}^2} - \frac{q_1}{(1+h)} + \frac{q_0{}^2 q_1}{C_1} - 2q_0(1+h)\cos C_5\left[\frac{3}{C_1}\right]^{1/2}$$

$$+ \frac{2q_0{}^3}{C_1{}^2}(3C_1)^{1/2}(1+h)^2\cos C_5 \tag{C.27}$$

where $$q_o = \left[\frac{2}{C_1}(1+h) - C_2(1+h)^2\right]^{-1/2} \tag{C.8}$$

Eq (C.8) can be substituted into Eq (C.27) to produce a complicated integral with h. However, for typical lifting earth atmospheric entry, altitudes of interest range from a minimum of 0 to a maximum of about 100 nautical miles. This is equivalent to a range of non-dimensional altitude of about $0.0 \leq h \leq 0.029$. The maximum h encountered for a typical lifting entry trajectory for Earth is thus always less than the small parameter, ϵ, allowing h to also be utilized as a small parameter in the outer expansion differential equations. The use of h as a small parameter considerably simplifies the derivation of solutions to some of the first order ϵ differential equations.

Expanding the terms in Eq (C.27) for small h and neglecting the resultant terms of $O(h^2)$ greatly simplifies the analytical integration of this equation. One term that appears often in Eq (C.27) is q_0. Rewriting Eq (C.8)

$$q_o = \left[\frac{2}{C_1} + \frac{2h}{C_1} - C_2 - h^2 C_2 - 2C_2 h\right]^{-1/2}$$

A condition allowing the expansion of this equation for small h is given by

$$\frac{2}{C_1}(1+h) > C_2(1+h)^2$$

This condition can be easily seen to always hold true for any values of C_1, C_2, and h.

Expanding Eq (C.8) for small h using a Taylors series approximation gives

$$q_o = \left[\frac{2}{C_1} - C_2\right]^{-1/2} - h\left[\frac{1}{C_1} - C_2\right]\cdot\left[\frac{2}{C_1} - 2C_2\right]^{-3/2} + O(h^2) \quad (C.28)$$

Some other terms appearing in Eq (C.27) are likewise expanded:

$$q_o(1+h) = \left[1 + \frac{h}{2 - C_1 C_2}\right]\cdot\left[\frac{C_1}{2 - C_1 C_2}\right]^{-1/2} + O(h^2) \quad (C.29)$$

$$q_0^{\ 2} = \left[\frac{2}{C_1} - C_2\right]^{-1} - h\left[\frac{1}{C_1} - C_2\right] / \left[\frac{2}{C_1} - C_2\right]^2 + O(h^2) \tag{C.30}$$

$$q_0^{\ 3} = \left[\frac{2}{C_1} - C_2\right]^{-3/2} - 3h\left[\frac{1}{C_1} - C_2\right] / \left[\frac{2}{C_1} - C_2\right]^{5/2} + O(h^2) \tag{C.31}$$

Defining two new constants, both dependent upon C_1 and C_2, simplifies Eqs (C.28) - (C.31).

$$\text{let} \quad C_{11} = \left[\frac{2}{C_1} - C_2\right] \quad \text{and} \quad C_{12} = \left[\frac{1}{C_1} - C_2\right] \tag{C.32}$$

With substitution, Eqs (C.28) - (C.31) become

$$q_o = (C_{11})^{-1/2} - \frac{C_{12}h}{(C_{11})^{1/2}} + O(h^2) \tag{C.33}$$

$$q_o(1+h) = (C_{11})^{-1/2} + \frac{h}{C_1(C_{11})^{3/2}} + O(h^2) \tag{C.34}$$

$$q_0^{\ 2} = \frac{1}{C_{11}} - \frac{2C_{12}h}{(C_{11})^2} + O(h^2) \tag{C.35}$$

$$q_0^{\ 3} = (C_{11})^{-3/2} - \frac{3C_{12}h}{(C_{11})^{5/2}} + O(h^2) \tag{C.36}$$

Two other terms in Eq (C.27) can also be expanded.

$$q_0^{\ 3}(1+h)^2 = (C_{11})^{-3/2} + \frac{h}{(C_{11})^{3/2}}\left[2 - \frac{3C_{12}}{C_{11}}\right] + O(h^2) \tag{C.37}$$

$$\frac{-q_1}{(1+h)} = -q_1 + q_1 h + O(h^2) \tag{C.38}$$

Substitution of Eqs (C.33) - (C.38) into Eq (C.27) gives a solvable expression for dq_1/dh.

$$\frac{dq_1}{dh} = q_1 C_{16} + q_1 C_{15} h + C_{14} h + C_{13} \tag{C.39}$$

where

$$C_{16} = \frac{2}{C_1 (C_{11})^{1/2}} + \frac{1}{C_1 C_{11}} - 1 \tag{C.40}$$

$$C_{15} = 1 - \frac{2C_{12}}{C_1 C_{11}} \cdot \left[\frac{1}{C_{11}} + \frac{1}{(C_{11})^{1/2}}\right] \tag{C.41}$$

$$C_{14} = \frac{3C_6 C_{12}}{(C_1)^2 (C_{11})^{5/2}}$$

$$+ 2\cos(C_5)\left[1 - 3\frac{C_{12}}{C_{11}}\right] \cdot \left[\frac{3}{C_1{}^3 C_{11}{}^3}\right]^{1/2} \tag{C.42}$$

$$C_{13} = 2\cos(C_5)\left[\frac{1}{C_1 C_{11}} - 1\right] \cdot \left[\frac{3}{C_1 C_{11}}\right]^{1/2} - \frac{3C_6}{C_1{}^2 C_{11}{}^{3/2}} \tag{C.43}$$

The differential equation given by Eq (C.39) is in the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

This form gives the following solution

$$y \cdot \exp\left[\int P(x)dx\right] = \int Q(x)\left[\int P(x)dx\right]dx + C$$

For the dq_1/dh differential equation $y = q_1$, $x = h$,

$$P(h) = -C_{16} - C_{15}h \ , \qquad \text{and} \qquad Q(h) = C_{13} + C_{14}h$$

With substitution

$$\exp\left[\int P(h)\,dh\right] = \exp\left[-C_{16}h - \frac{1}{2}(C_{15})h^2\right] = \exp\left[-C_{16}h + O(h^2)\right]$$

Therefore

$$q_1 \exp\left[-C_{16}h\right] = C_{13}\int \exp\left[-C_{16}h\right]dh + C_{14}\int h\left[\exp(-C_{16}h)\right]dh + C_{20}$$

and the solution to the dq_1/dh equation is given by

$$q_1 = -\frac{C_{13}}{C_{16}} - \frac{C_{14}}{(C_{16})^2} - h\left[\frac{C_{14}}{C_{16}}\right] - C_{20}\exp\left[-C_{16}h\right] \qquad (C.44)$$

The constants C_{13}, C_{14}, and C_{16} are given by Eqs (C.43), (C.42), and (C.40), respectively. Because h^2 terms are neglected, C_{15} is dropped from the solution. Hence, a solution for q_1 has been found as a function of two constants of integration, C_6 and C_{20}, and the constants C_{13}, C_{14}, and C_{16}, which are dependent upon C_1 and C_2 from the relations given in Eq (C.32).

Appendix D

Derivation of Solutions to Selected Inner Expansion Differential Equations

Inner Expansion ϵ^0 Terms

The ϵ^0 terms of the inner expansions found in Section V are as follows:

$$\frac{du_0}{d\xi} = -\ 2u_0 Be^{-\xi}\left[\frac{1}{\sin\gamma_0} + \frac{C_L\cos\sigma}{C_D\cos\gamma_0}\right] \tag{D.1}$$

$$\frac{dq_0}{d\xi} = -\ \frac{C_L}{C_D}\cdot Be^{-\xi}\cos\sigma \tag{D.2}$$

$$\frac{dI_0}{d\xi} = \frac{C_L}{C_D}\cdot Be^{-\xi}\cdot\frac{\sin\sigma\cos\alpha_0}{\cos\gamma_0\sin\gamma_0} \tag{D.3}$$

$$\frac{d\Omega_0}{d\xi} = \frac{C_L}{C_D}\cdot Be^{-\xi}\cdot\frac{\sin\sigma\sin\alpha_0}{\cos\gamma_0\sin\gamma_0\sin I_0} \tag{D.4}$$

$$\frac{d\alpha_0}{d\xi} = -\ \frac{C_L}{C_D}\cdot Be^{-\xi}\cdot\frac{\sin\sigma\sin\alpha_0}{\tan I_0\sin\gamma_0\cos I_0} \tag{D.5}$$

Solutions to this set of differential equations are derived below.

<u>dq/dξ Equation Solution</u>. From Eq (D.2)

$$\frac{dq_0}{d\xi} = -\ \frac{C_L}{C_D}\cdot Be^{-\xi}\cos\sigma$$

This is a very simple integration by separation of variables. The constant of integration is K_1.

$$q_0 = \frac{C_L}{C_D} \cdot Be^{-\xi}\cos\sigma + K_1 \tag{D.6}$$

<u>du/dh Equation Solution</u>. From Eq (D.1)

$$\frac{du_0}{d\xi} = -\ 2u_0\,Be^{-\xi} \cdot \left[\frac{1}{\sin\gamma_0} + \frac{C_L}{C_D}\cdot\frac{\cos\sigma}{\cos\gamma_0}\right]$$

Previously it was seen that $q_0 = \cos\gamma_0$

Therefore, $\dfrac{dq_0}{d\xi} = -\sin\gamma_0 \dfrac{d\gamma_0}{d\xi}$ and $\dfrac{d\gamma_0}{d\xi} = \dfrac{C_L}{C_D}Be^{-\xi}\ \dfrac{\cos\sigma}{\sin\gamma_0}$

The du/dh equation can be rewritten

$$\int \frac{1}{u_0}du_0 = \int\left[-2u_0\,Be^{-\xi}\frac{d\xi}{\sin\gamma_0}\right]\frac{d\gamma_0}{d\xi}\cdot\frac{d\xi}{d\gamma_0}$$

$$+ \int\left[-2u_0\,Be^{-\xi}\ \frac{C_L}{C_D}\cdot\frac{\cos\sigma}{q_0}d\xi\right]\frac{dq_0}{d\xi}\cdot\frac{d\xi}{dq_0}$$

$$\ln(u_0) = \int\left[-2u_0\,Be^{-\xi}\frac{1}{\sin\gamma_0}\right]^{+1} \cdot \left[\frac{C_L}{C_D}\cdot Be^{-\xi}\ \frac{\cos\sigma}{\sin\gamma_0}\right]^{-1} d\gamma_0$$

$$+ \int\left[-2u_0\,Be^{-\xi}\ \frac{C_L}{C_D}\cdot\frac{\cos\sigma}{q_0}\right]^{+1} \cdot \left[\frac{C_L}{C_D}\cdot Be^{-\xi}\cos\sigma\right]^{-1} dq_0$$

After reducing, this equation takes the form

$$\ln(u_0) = \int\frac{-C_D 2d\gamma_0}{C_L\cos\sigma} + \int\frac{2}{q_0}\,dq_0 = \frac{-C_D 2\gamma_0}{C_L\cos\sigma} + 2\ln(\gamma_o) + \ln(K_3)$$

Solution: $$u_0 = K_3\,q_0{}^2\exp\left[\frac{-C_D 2\gamma_0}{C_L\cos\sigma}\right] \tag{D.7}$$

<u>dI/dξ and dα/dξ Equations</u>. From Eq (D.3)

$$\frac{dI_0}{d\xi} = \frac{C_L}{C_D} \cdot Be^{-\xi} \cdot \frac{\sin\sigma\cos\alpha_0}{\cos\gamma_0 \sin\gamma_0}$$

From Eq (D.5) $$\frac{d\alpha_0}{d\xi} = -\frac{C_L}{C_D} \cdot Be^{-\xi} \cdot \frac{\sin\sigma\sin\alpha_0}{\tan I_0 \sin\gamma_0 \cos I_0}$$

By the chain rule $$\frac{dI_0}{d\alpha_0} = \frac{dI_0}{d\xi} \cdot \frac{d\xi}{d\alpha_0} = -\frac{\tan I_0}{\tan\alpha_0}$$

With separation of variables, this integration simply produces

$$\sin\alpha_0 \sin I_0 = \sin K_4 \tag{D.8}$$

or

$$I_0 = \sin^{-1}\left[\frac{\sin K_4}{\sin\alpha_0}\right] \tag{D.9}$$

<u>$d\alpha/d\xi$ and $d\Omega/d\xi$ Equations</u>. From Eq (D.5)

$$\frac{d\alpha_0}{d\xi} = -\frac{C_L}{C_D} \cdot Be^{-\xi} \cdot \frac{\sin\sigma\sin\alpha_0}{\tan I_0 \sin\gamma_0 \cos I_0}$$

From Eq (D.4) $$\frac{d\Omega_0}{d\xi} = \frac{C_L}{C_D} \cdot Be^{-\xi} \cdot \frac{\sin\sigma\sin\alpha_0}{\cos\gamma_0 \sin\gamma_0 \sin I_0}$$

Chain rule $$\frac{d\alpha_0}{d\Omega_0} = \frac{d\alpha_0}{d\xi} \cdot \frac{d\xi}{d\Omega_0} = -\cos I_0$$

Substitution gives

$$\frac{d\alpha_0}{d\Omega_0} = -\cos\left[\sin^{-1}\left[\frac{\sin K_4}{\sin\alpha_0}\right]\right] = -\left[1 - \left[\frac{\sin K_4}{\sin\alpha_0}\right]^2\right]^{1/2}$$

Rewriting this equation

$$\int d\Omega = \int \frac{-\sin\alpha_0 \; d\alpha_0}{\left[\sin^2\alpha_0 - \sin^2 K_4\right]^{1/2}}$$

To solve this integral, let $x = \cos\alpha_0$ and

$a = 1 - \sin^2 K_4 = \cos^2 K_4$,

so $dx = -\sin\alpha_0 \cdot d\alpha_0$ and $1 - x^2 = \sin^2\alpha_0$

Therefore, the integral can be written in the form

$$\int \frac{dx}{\left[a^2 - x^2\right]^{1/2}} = -\cos^{-1}\left[\frac{x}{|a|}\right]$$

Solution: $$\Omega_0 = -\cos^{-1}\left[\frac{\cos\alpha_0}{\cos K_4}\right] - K_5 \qquad \text{(D.10)}$$

or $$\cos\alpha_0 = \cos K_4 \cos(K_5 - \Omega_0) \qquad \text{(D.11)}$$

dI/dξ and dγ/dξ Equations. From Eq (D.3)

$$\frac{dI_0}{d\xi} = \frac{C_L}{C_D} \cdot Be^{-\xi} \sin\sigma \frac{\cos\alpha_0}{\cos\gamma_0 \sin\gamma_0}$$

From above $$\frac{d\gamma_0}{d\xi} = \frac{C_L}{C_D} \cdot Be^{-\xi} \frac{\cos\sigma}{\sin\gamma_0}$$

By the chain rule $$\frac{dI_0}{d\gamma_0} = \frac{dI_0}{d\xi} \cdot \frac{d\xi}{d\gamma_0} = \tan\sigma \cdot \frac{\cos\alpha_0}{\cos\gamma_0}$$

Since $\alpha_0 = \sin^{-1}\left[\frac{\sin K_4}{\sin I_0}\right]$ then by substitution

$$\tan\sigma\int\frac{d\gamma_0}{\cos\gamma_0} = \int\left[\cos\left[\sin^{-1}\left[\frac{\sin K_4}{\sin I_0}\right]\right]\right]^{-1} dI_0$$

This can be rewritten as

$$\tan\sigma\int\frac{d\gamma_0}{\cos\gamma_0} = \int\left[1 - \frac{\sin^2 K_4}{\sin^2 I_0}\right]^{-1/2} dI_0$$

$$= \int\frac{\sin I_0}{\left[\sin^2 K_4 - 1 + \sin^2 I_0\right]^{1/2}}\cdot dI_0$$

To solve this integral, let $x = \cos I_0$ and

$a^2 = 1 - \sin^2 K_4 = \cos^2 K_4$

so $dx = -\sin I_0\, dI_0$ and

$1 - x^2 = \sin^2 I_0$

Thus, the right side of the integral can be written in the form

$$\int\frac{dx}{\left[a^2 - x^2\right]^{1/2}} = \cos^{-1}\left[\frac{x}{|a|}\right] = \cos^{-1}\left[\frac{\cos I_0}{\cos K_4}\right]$$

and the left side of the integral is

$$\tan\sigma\int\frac{d\gamma_0}{\cos\gamma_0} = \tan\sigma\log\left[\tan\left[\frac{\pi}{4} + \frac{\gamma_0}{2}\right]\right] + K_6$$

Solution:

$$\cos I_0 = \cos K_4 \cos\left[\tan\sigma\cdot\log\left[\tan\left(\frac{\pi}{4} + \frac{\gamma_0}{2}\right)\right] + K_6\right] \tag{D.12}$$

Inner Expansion ϵ^1 Terms

The ϵ^1 terms of the outer expansions found in Section V are as follows:

$$\frac{dI_1}{d\xi} = \frac{C_L}{C_D}\cdot Be^{-\xi}\sin\sigma\left[\frac{-\alpha_1\sin\alpha_0}{\cos\gamma_0\sin\gamma_0} + \gamma_1\frac{\cos\alpha_0}{\cos^2\gamma_0} - \gamma_1\frac{\cos\alpha_0}{\sin^2\gamma_0}\right] \tag{D.13}$$

$$\frac{d\alpha_1}{d\xi} = -\frac{\tan\alpha_0}{\tan I_0}\cdot\frac{dI_1}{d\xi} \tag{D.14}$$

or $$\frac{d\alpha_1}{d\xi} = \frac{C_L}{C_D}\cdot Be^{-\xi}\sin\sigma\left[\gamma_1\frac{\sin\alpha_0}{\cos^2\gamma_0\tan I_0} - \gamma_1\frac{\sin\alpha_0}{\sin^2\gamma_0\tan I_0}\right.$$

$$\left. - I_1\frac{\sin\alpha_0}{\cos\gamma_0\sin\gamma_0\sin^2 I_0} + \alpha_1\frac{\cos\alpha_0}{\cos\gamma_0\sin\gamma_0\tan I_0}\right]$$

$$\frac{du_1}{d\xi} = 2u_0 Be^{-\xi}\cdot\left[\gamma_1\frac{\cos\gamma_0}{\sin^2\gamma_0} + \gamma_1\frac{C_L}{C_D}\cdot\frac{\cos\sigma}{\cos\gamma_0} + \frac{C_L}{C_D}\cdot\frac{\gamma_1\cos\sigma}{\sin\gamma_0\cos^2\gamma_0}\right]$$

$$- 2u_1 Be^{-\xi}\cdot\left[\frac{1}{\sin\gamma_0} + \frac{C_L\cos\sigma}{C_D\cos\gamma_0}\right] \tag{D.15}$$

$$\frac{d\Omega_1}{d\xi} = \frac{\tan\alpha_0}{\sin I_0}\cdot\frac{dI_1}{d\xi}$$

or $$\frac{d\Omega_1}{d\xi} = \frac{C_L}{C_D}\cdot Be^{-\xi}\sin\sigma\left[\gamma_1\frac{\sin\alpha_0}{\cos^2\gamma_0\sin I_0} - \gamma_1\frac{\sin\alpha_0}{\sin^2\gamma_0\sin I_0}\right.$$

$$\left. - I_1\frac{\sin\sigma\sin\alpha_0}{\cos\gamma_0\sin\gamma_0\sin^2 I_0} + \gamma_1\frac{\cos\alpha_0}{\cos\gamma_0\sin\gamma_0\sin I_0}\right] \tag{D.16}$$

$$\frac{dq_1}{d\xi} = 0 \tag{D.17}$$

To solve the dq_1/dh equation is a trivial exercise; solutions to Eqs (D.13) - (D.16) are not required.

<u>$dq/d\xi$ Equation Solution</u>. From Eq (D.17), q_1 is simply

$$q_1 = K_7 \tag{D.18}$$

Appendix E

Example Earth Atmospheric Entry Trajectories

This appendix presents three example Earth atmospheric entry trajectories to give an indication of what the dimensionless variables u and h are in relation to more conventional variables such as Mach, altitude, flight path angle, and velocity. These trajectories were generated by AFWAL/FIMG, Air Force Flight Dynamics Laboratory personnel in the course of flight performance analysis support for current hypersonic entry vehicle studies. Modern trajectory analysis computer programs were utilized to produce this data.

Example Trajectory 1 is the gliding entry trajectory for a relatively high lift-to-drag ratio vehicle (about 3.0). Example Trajectory 2 is the gliding entry trajectory for a low lift-to-drag ratio vehicle (about 0.5). Example Trajectory 3 is a very steep gliding entry trajectory for the same low lift-to-drag ratio vehicle. This last trajectory is somewhat non-realistic due to the extremely high dynamic pressures (and hence severe heating) encountered. However, it is included to help give an indication of the extreme in u and h combinations. Many current and planned lifting entry vehicles have trajectories that fall within the ranges of u and h shown in these examples.

For the sample trajectories presented in Tables I, II, and III, the following variables are defined.

$y \equiv$ Altitude (kft)

$\gamma \equiv$ Flight path angle (deg)

$V \equiv$ Velocity (kft/s)

$h \equiv$ Non-dimensional altitude $= r/r_*$

$u \equiv$ Speed ratio

Table I Example Trajectory #1

Mach	y (kft)	γ (deg)	V (kft/s)	h (10^{-3})	u
28.3	300.0	-0.7	25.00	14.34	.942
27.3	252.5	-0.6	24.98	12.07	.939
27.0	270.5	-0.2	23.90	12.93	.860
24.6	240.0	+0.4	23.34	10.64	.819
21.4	212.7	+0.3	22.26	10.16	.744
18.3	200.0	-0.2	19.19	9.56	.553
17.4	201.4	-0.4	18.24	9.62	.499
13.0	176.4	-0.1	14.03	8.43	.295
10.9	160.7	-0.3	11.83	7.68	.210
9.1	150.6	0.0	9.75	7.20	.142
7.6	141.2	-0.1	7.99	6.75	.096
5.9	130.8	-0.5	6.13	6.25	.056
4.6	119.7	-0.8	4.70	5.72	.033
3.3	107.7	-0.7	3.29	5.15	.016
2.6	99.2	-2.1	2.54	4.74	.010
1.9	90.0	-4.0	1.87	4.30	.005
1.0	59.4	-11.0	0.97	2.84	.0014

Table II Example Trajectory #2

Mach	y (kft)	γ (deg)	V (kft/s)	h (10^{-3})	u
28.3	287.4	-0.4	25.01	13.73	.942
28.3	262.3	-0.7	25.01	12.53	.941
25.4	227.9	-0.9	24.92	10.89	.933
23.1	189.6	-0.8	24.47	9.06	.898
21.5	171.0	-0.5	23.24	8.17	.810
20.3	175.1	+0.5	21.93	8.37	.721
19.2	164.1	-0.7	20.73	7.84	.644
17.4	138.9	-0.5	18.29	6.64	.501
14.2	129.4	-0.5	14.78	6.18	.327
10.6	107.0	-1.2	10.53	5.11	.166
5.6	81.5	-3.3	5.48	3.90	.045
2.0	49.5	-3.8	1.97	2.40	.0085
1.2	25.7	-18.0	1.24	1.23	.0021

Table III Example Trajectory #3

Mach	y (kft)	γ (deg)	V (kft/s)	h (10^{-3})	u
28.3	300.0	-10.0	25.00	14.34	.914
27.7	256.5	-10.0	25.05	12.26	.916
24.6	212.8	-10.0	25.08	10.17	.916
23.1	169.3	-9.9	25.04	8.09	.912
22.8	92.7	-7.0	22.48	4.43	.743
19.1	72.3	-4.2	18.59	3.46	.513
11.1	66.6	+3.4	10.76	3.18	.172
7.7	89.2	+9.7	7.55	4.26	.083
6.3	125.	+3.8	6.45	5.97	.062
6.0	105.	-9.8	5.97	5.02	.052
5.1	71.1	-12.4	4.97	3.40	.035
2.2	47.7	+0.5	2.12	2.28	.007
1.6	50.1	-2.7	1.55	2.19	.0036

Appendix F

Supplemental Figures for Section VI

Supplemental figures for Section IV are presented on the following pages.

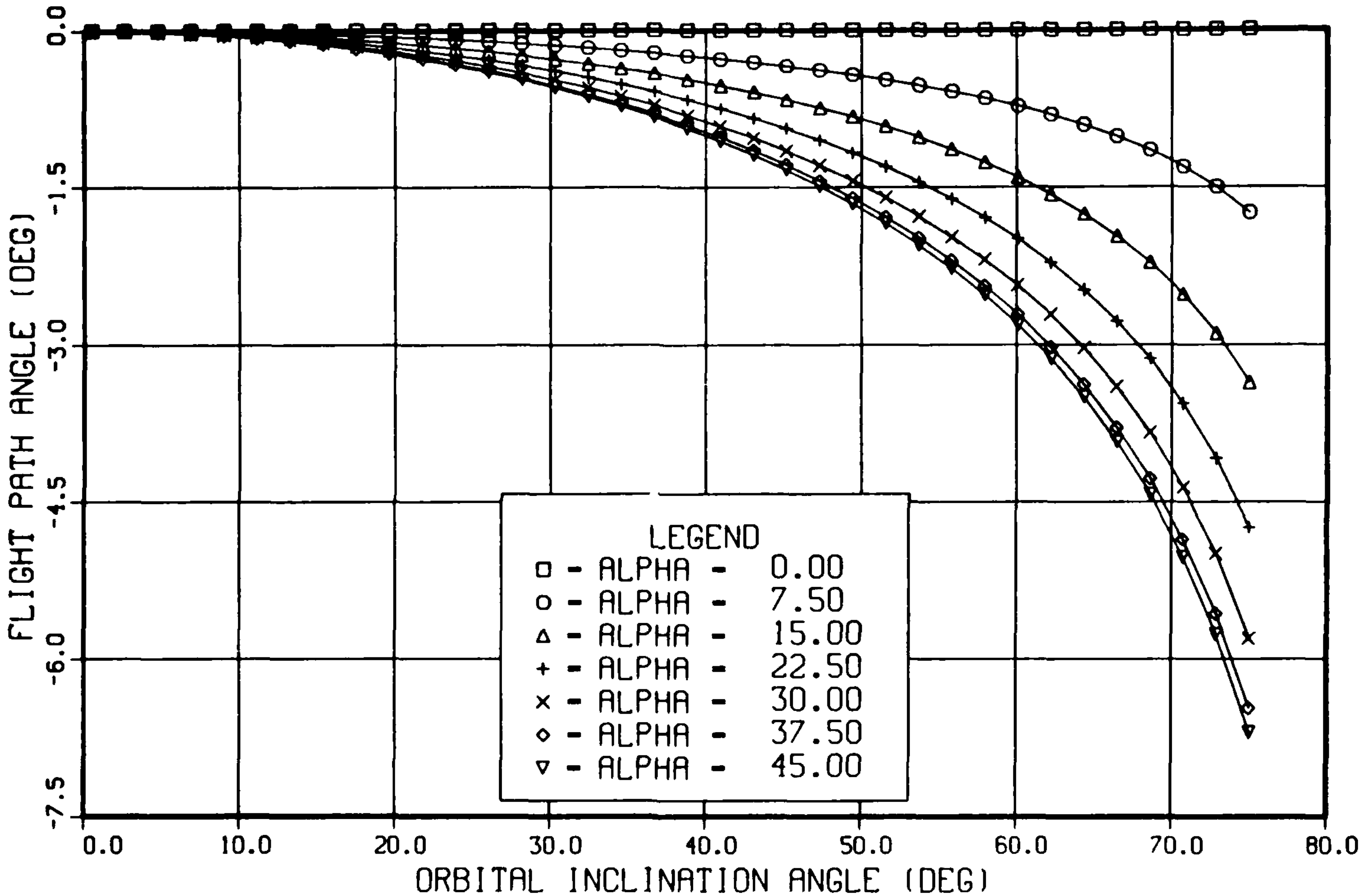

Figure F1. States of Validity for the Non-Rotating Earth, du/dh Equation of Motion (u = .210 , h = .0096)

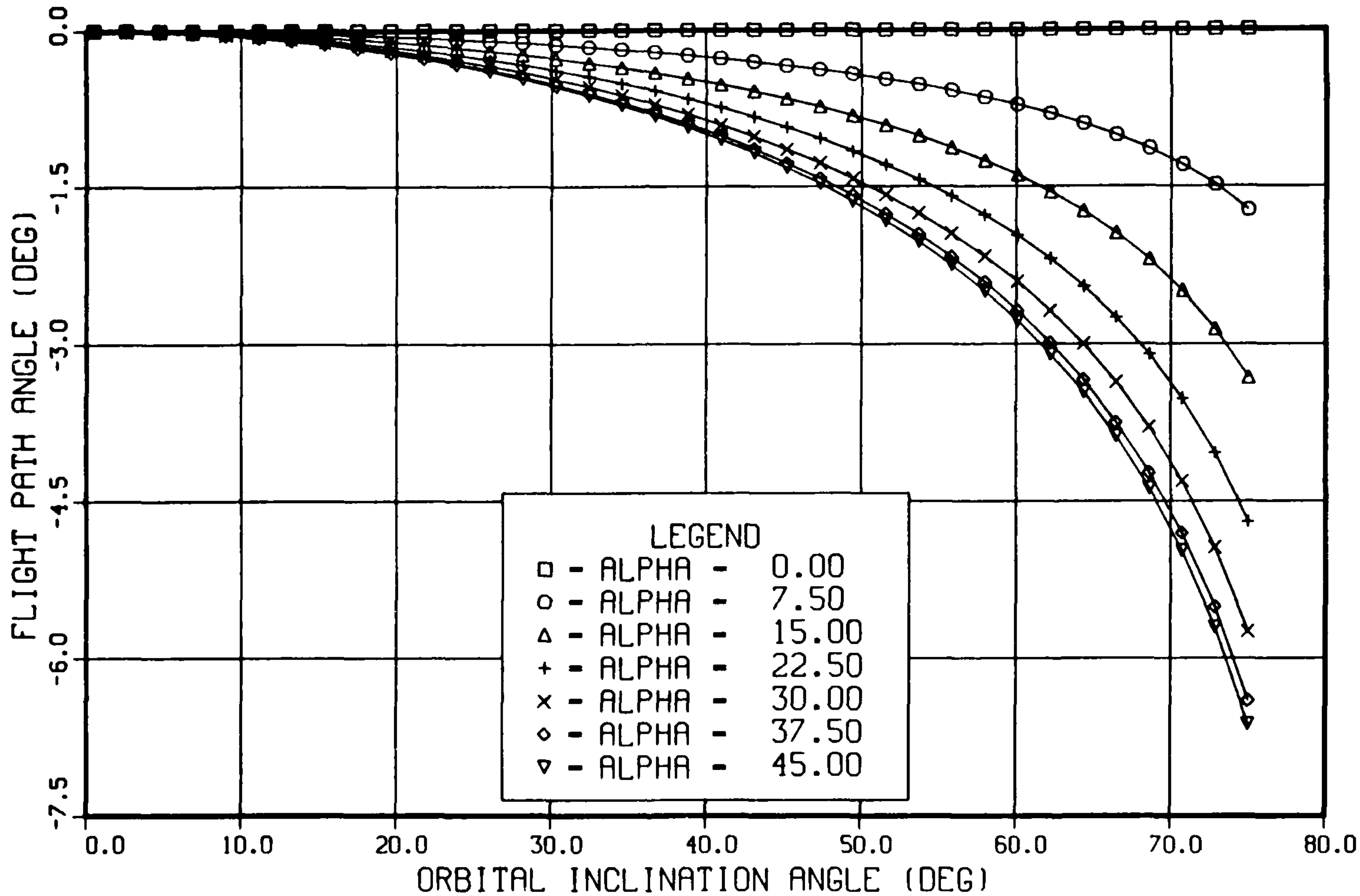

Figure F2. States of Validity for the Non-Rotating Earth, du/dh Equation of Motion ($u = .210$, $h = .0024$)

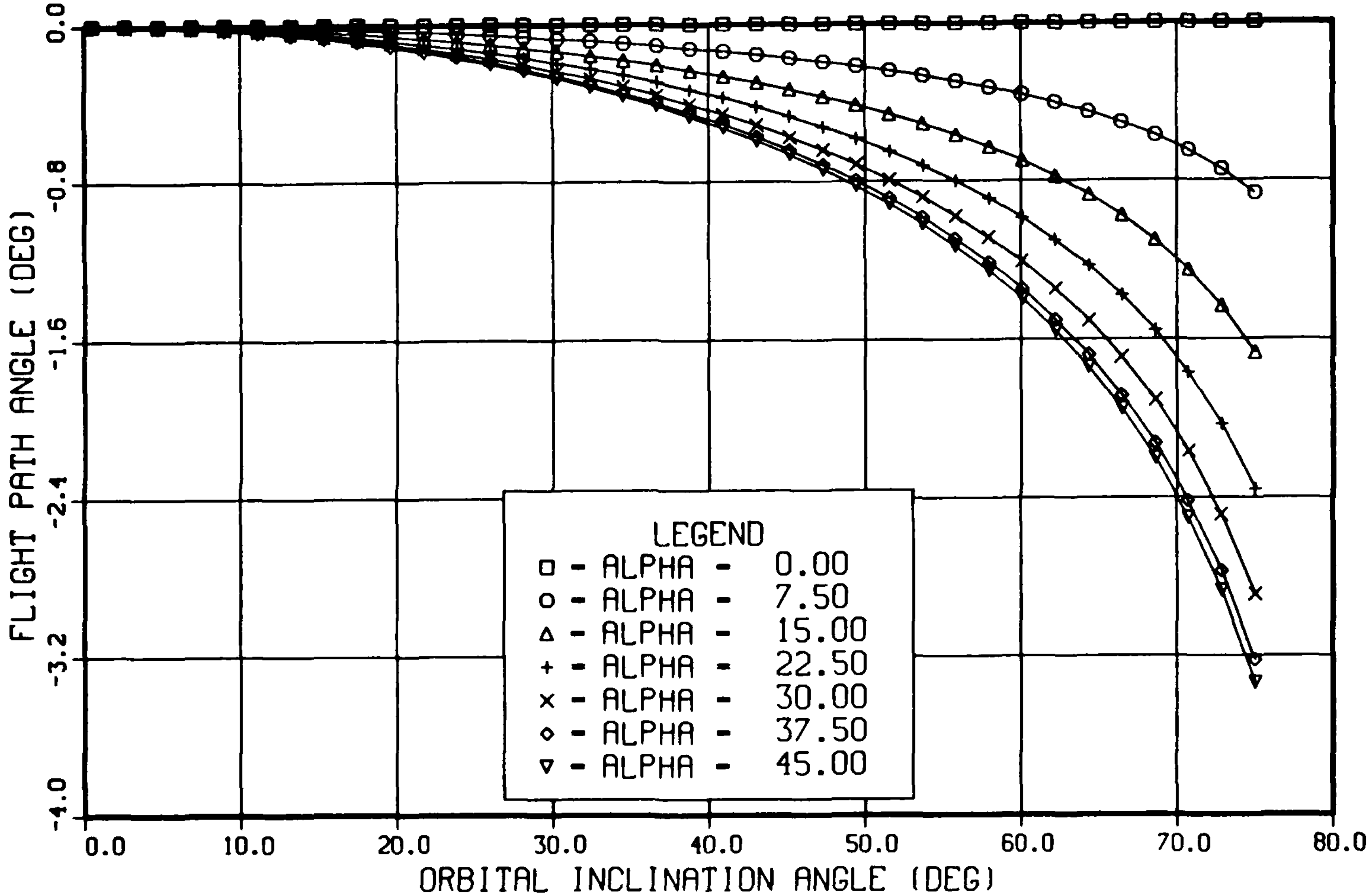

Figure F3. States of Validity for the Non-Rotating Earth, du/dh Equation of Motion (u = .860 , h = .0130)

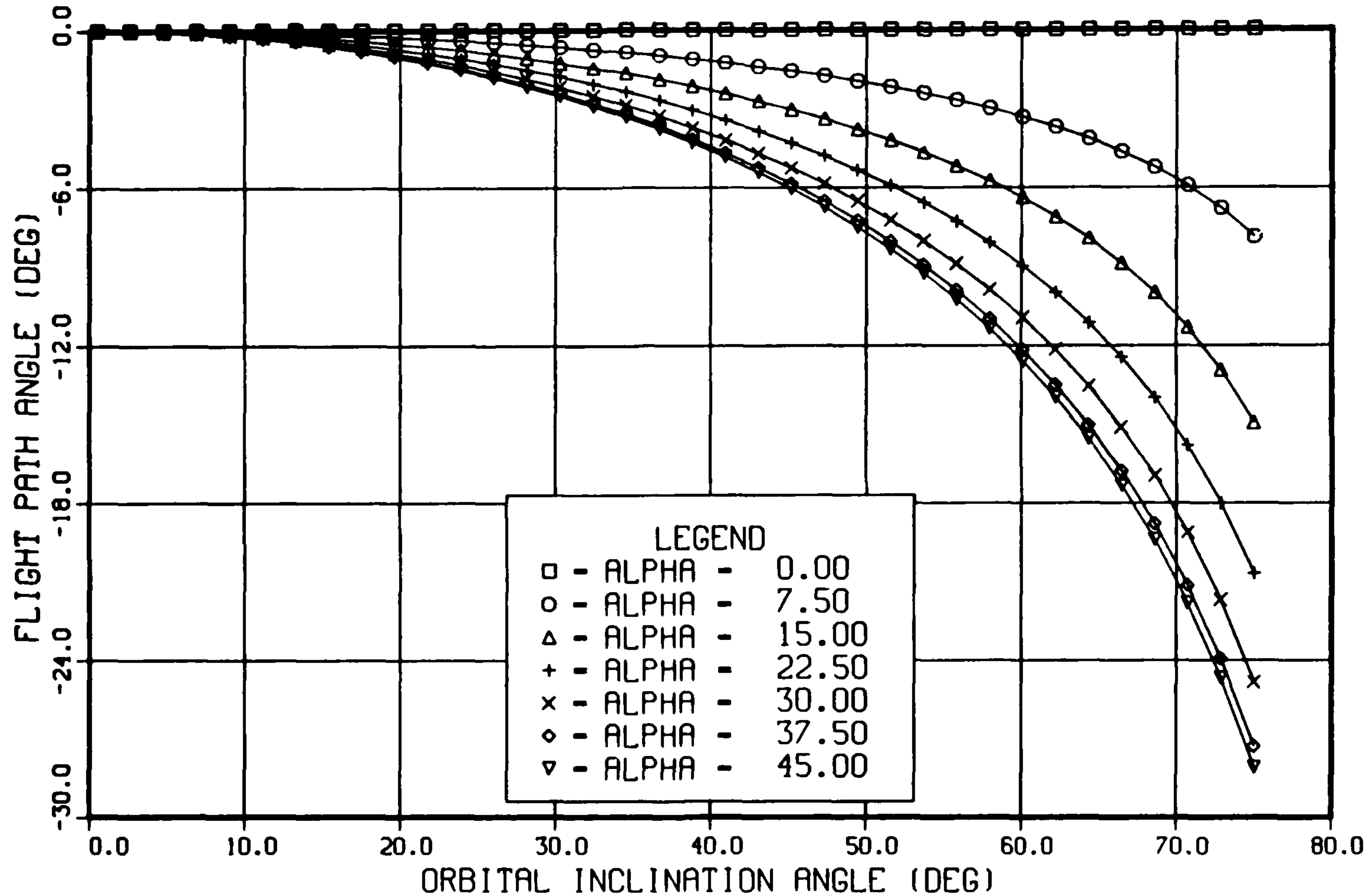

Figure F4. States of Validity for the Non-Rotating Earth, du/dh Equation of Motion (u = .010 , h = .0047)

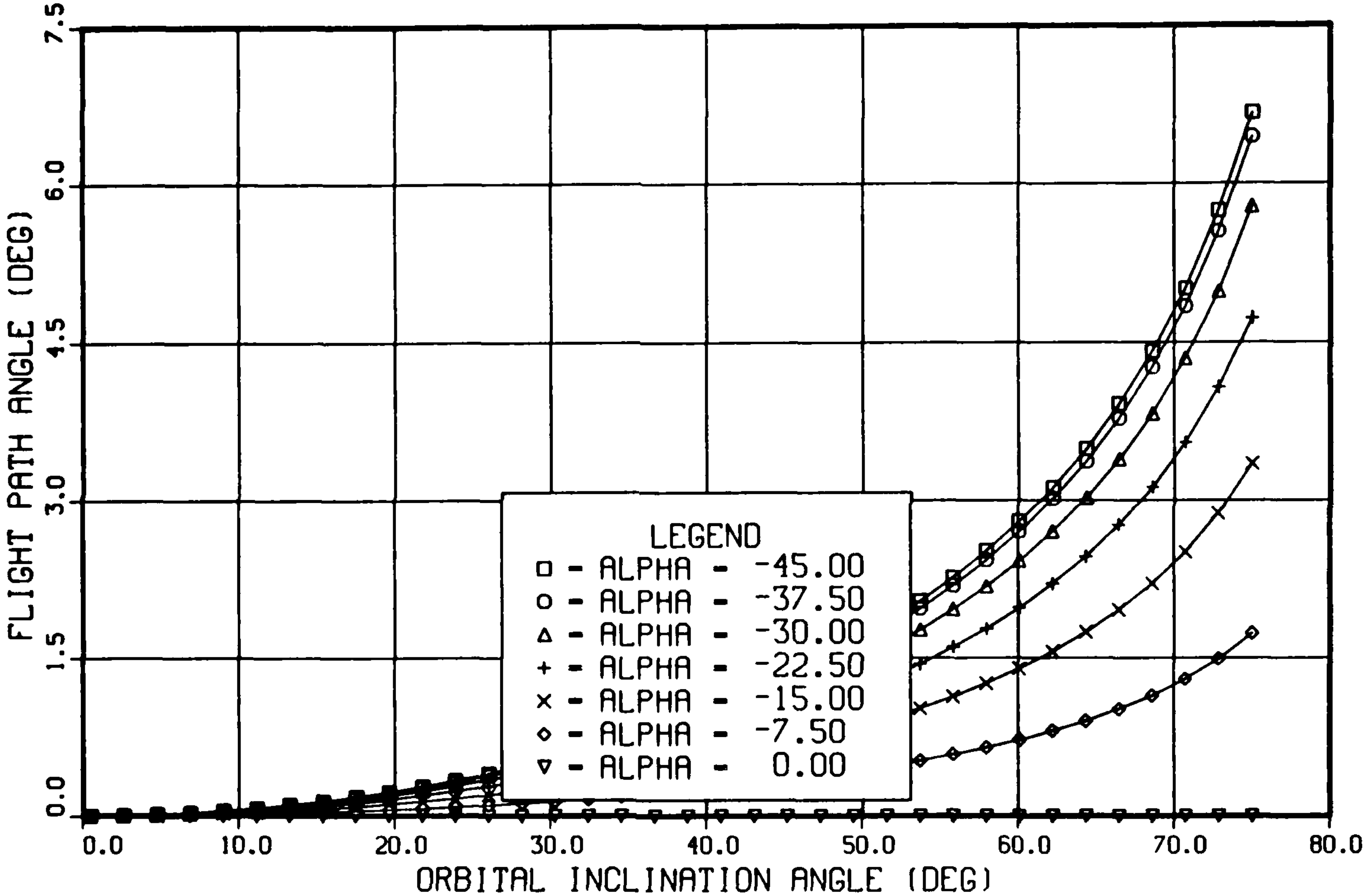

Figure F5. States of Validity for the Non-Rotating Earth, du/dh Equation of Motion (u = .210 , h = .0077)

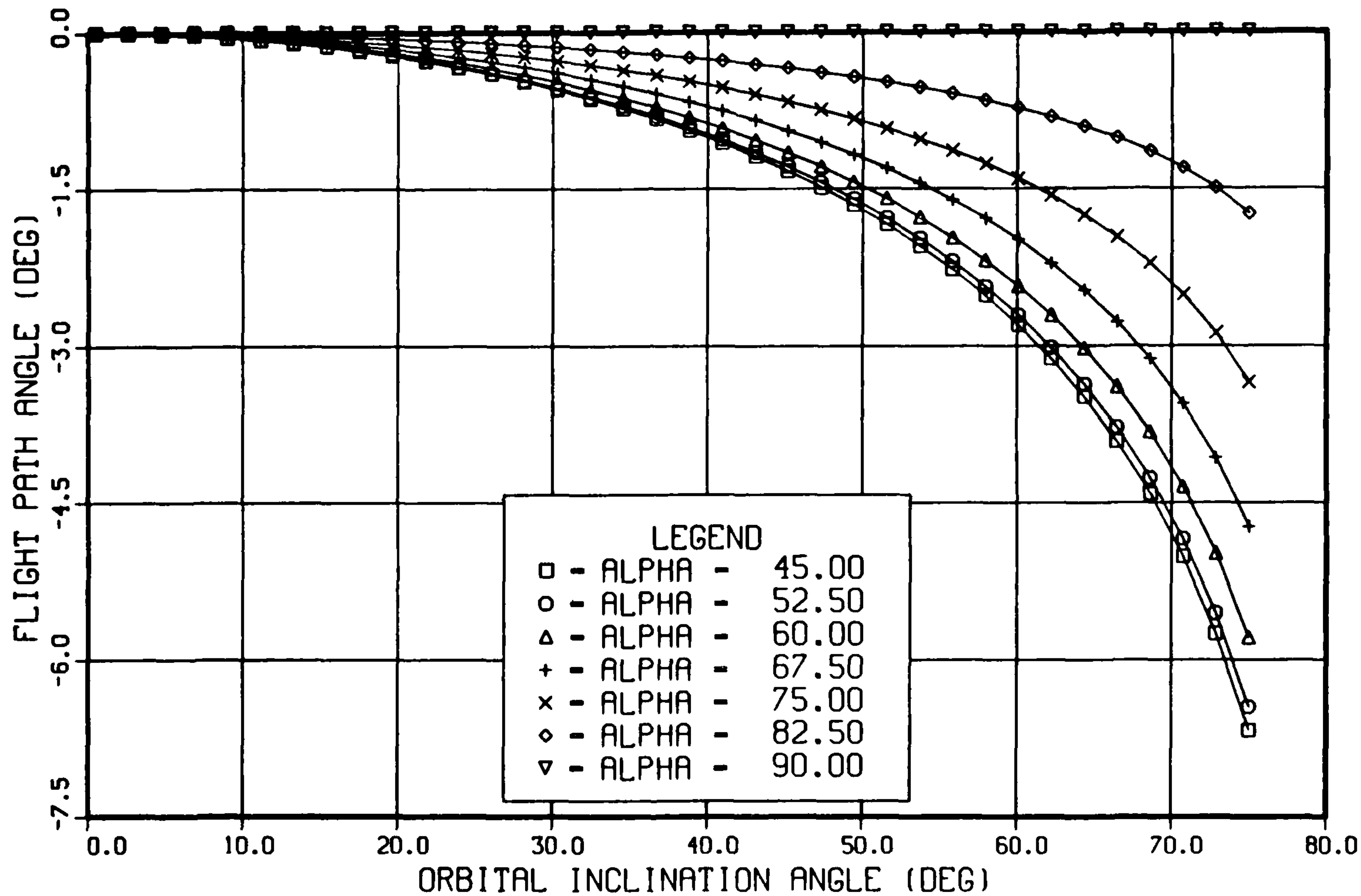

Figure F6. States of Validity for the Non-Rotating Earth, du/dh Equation of Motion (u = .210 , h = .0077)

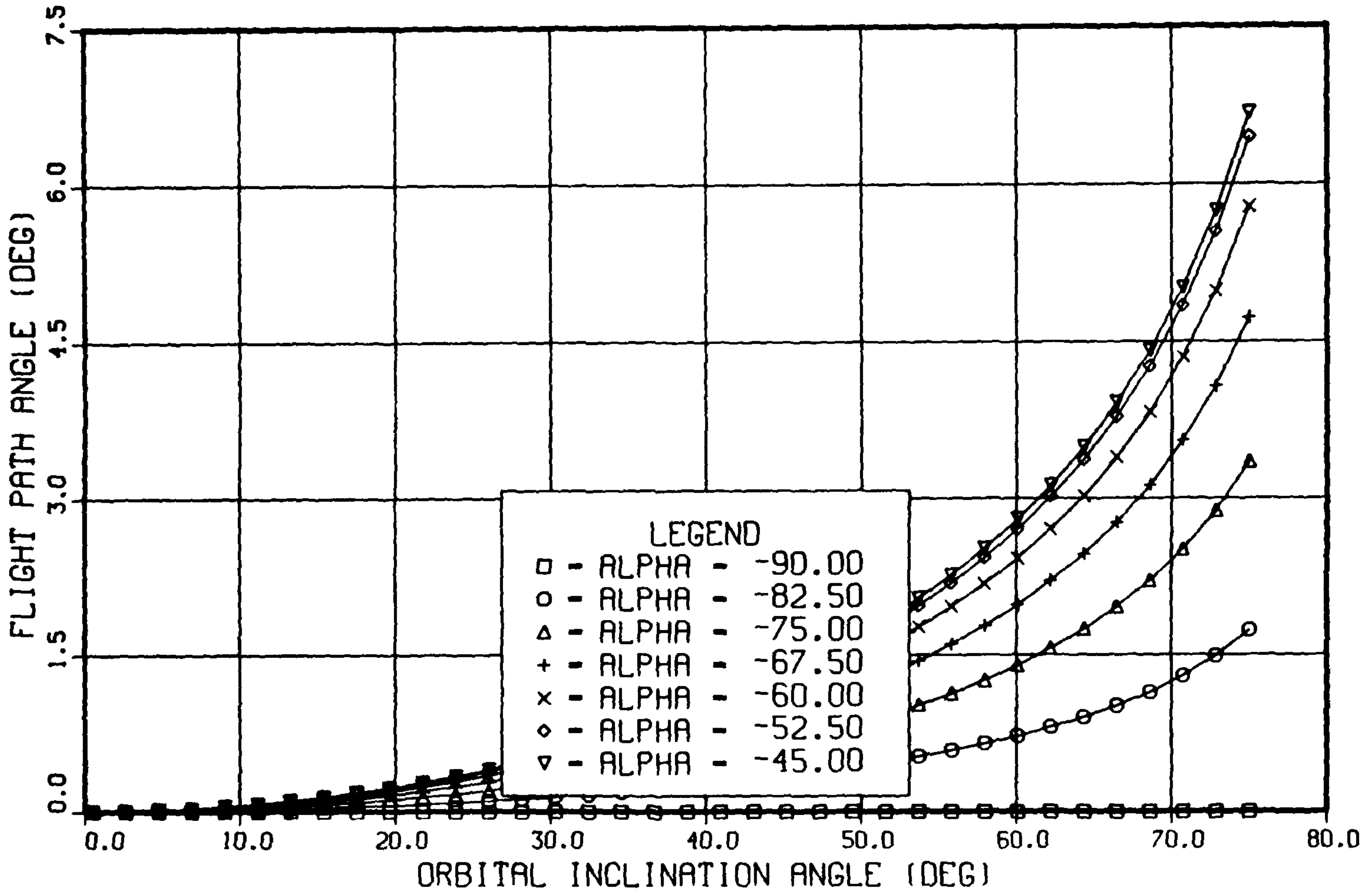

Figure F7. States of Validity for the Non-Rotating Earth, du/dh Equation of Motion (u = .210 , h = .0077)

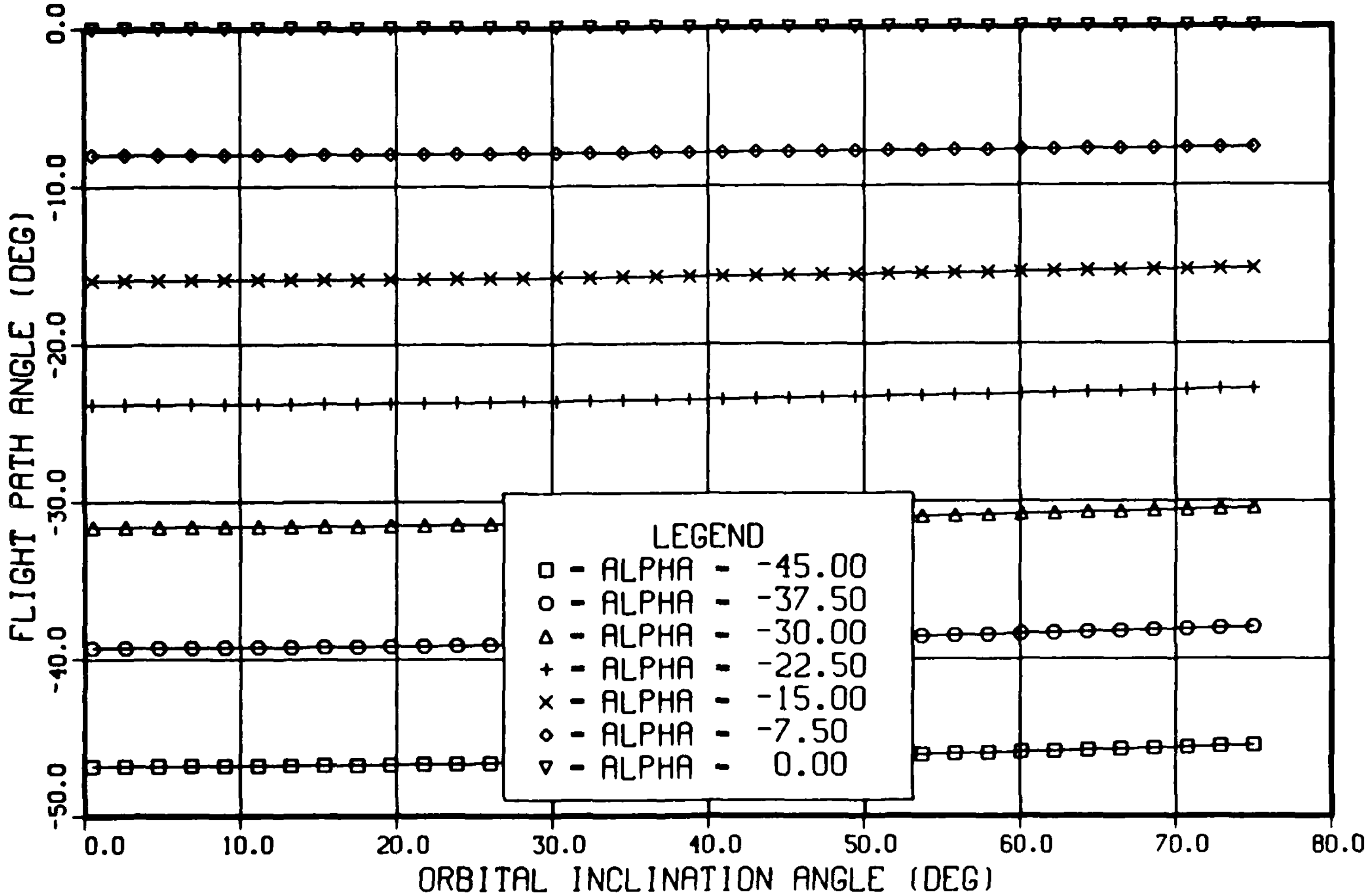

Figure F8. States of Validity for the Non-Rotating Earth, dI/dh Equation of Motion (u = .210 , h = .0096)

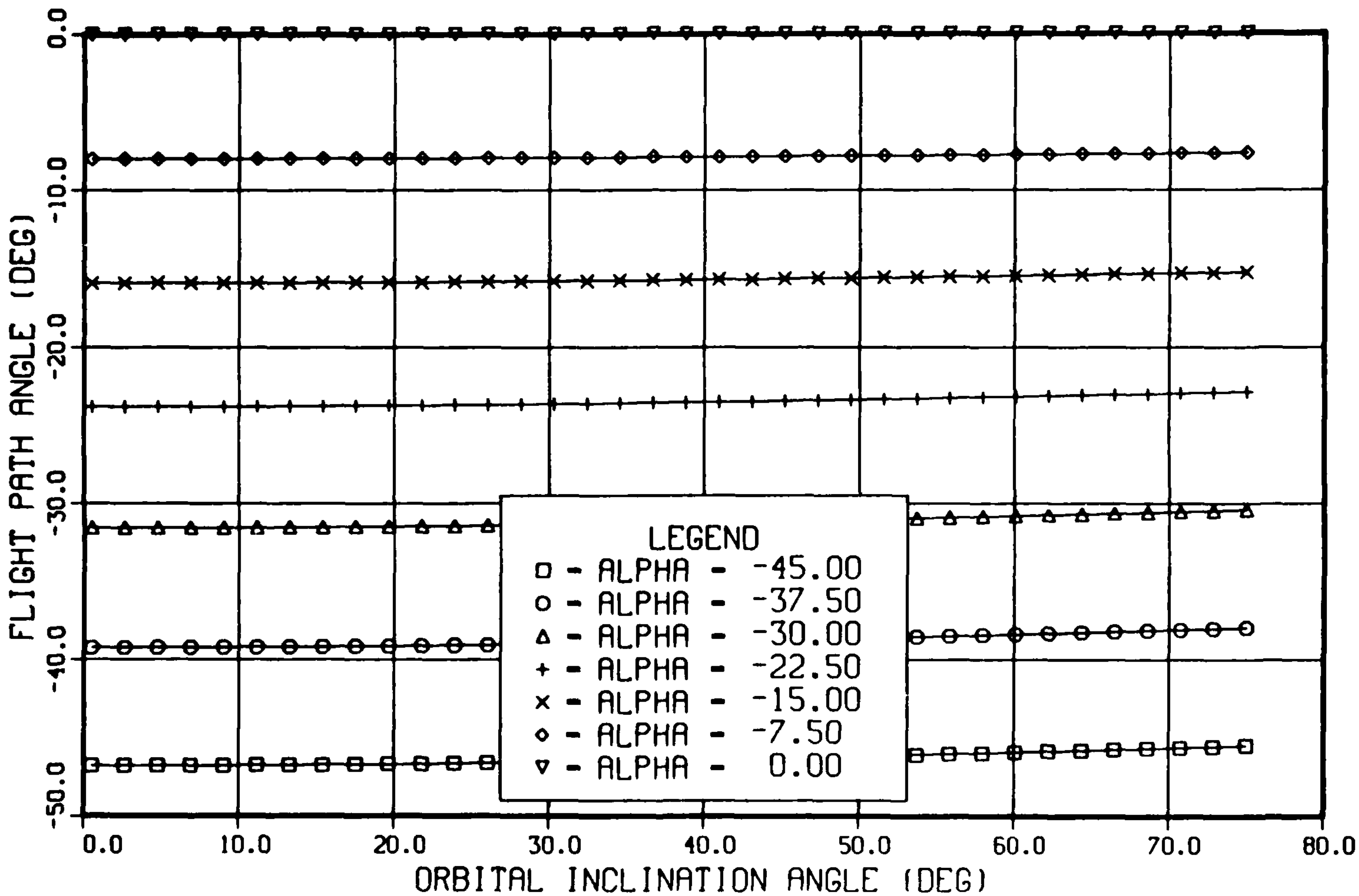

Figure F9. States of Validity for the Non-Rotating Earth, dI/dh Equation of Motion (u = .210 , h = .0024)

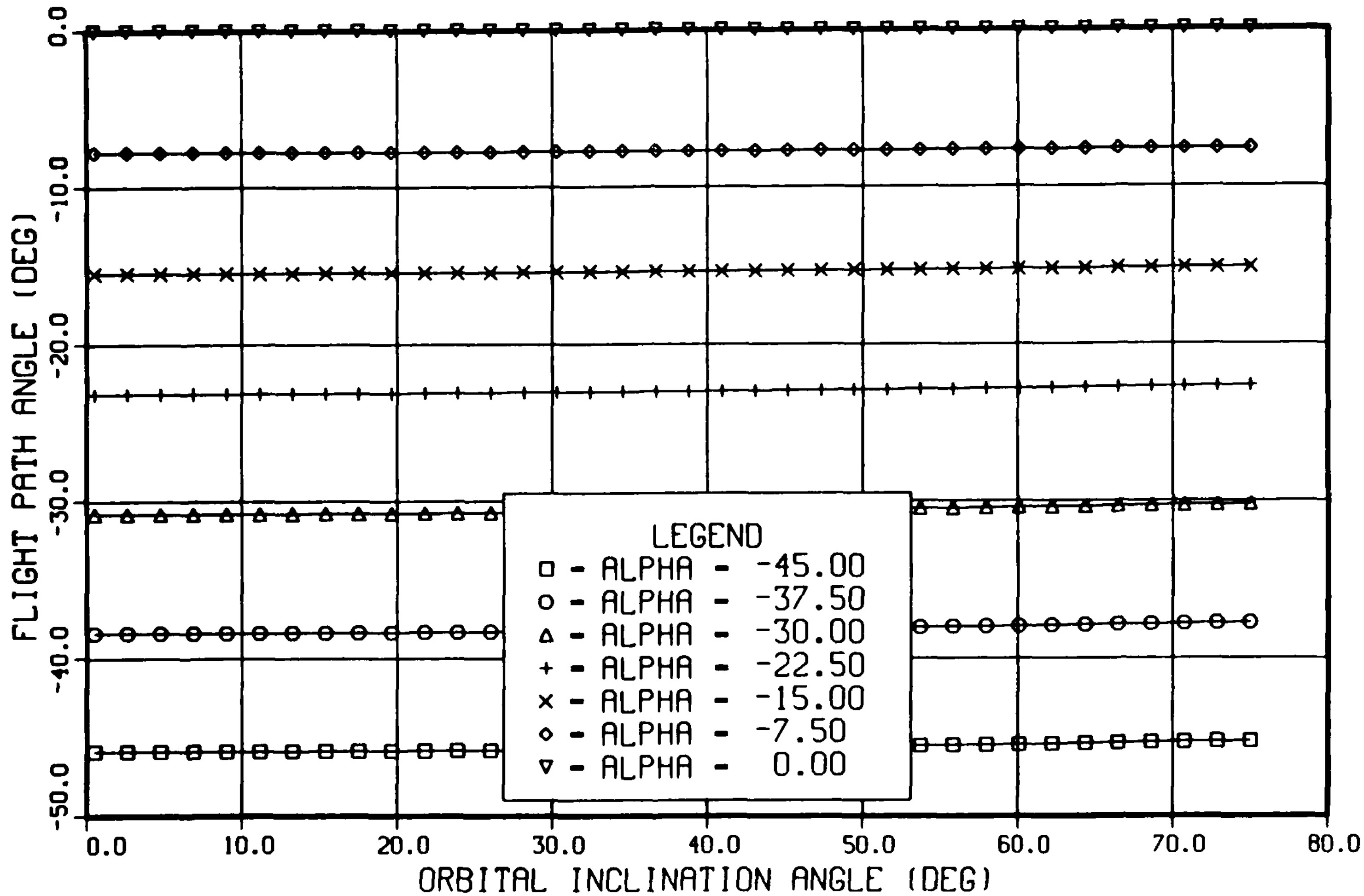

Figure F10. States of Validity for the Non-Rotating Earth, dI/dh Equation of Motion (u = .860 , h = .0160)

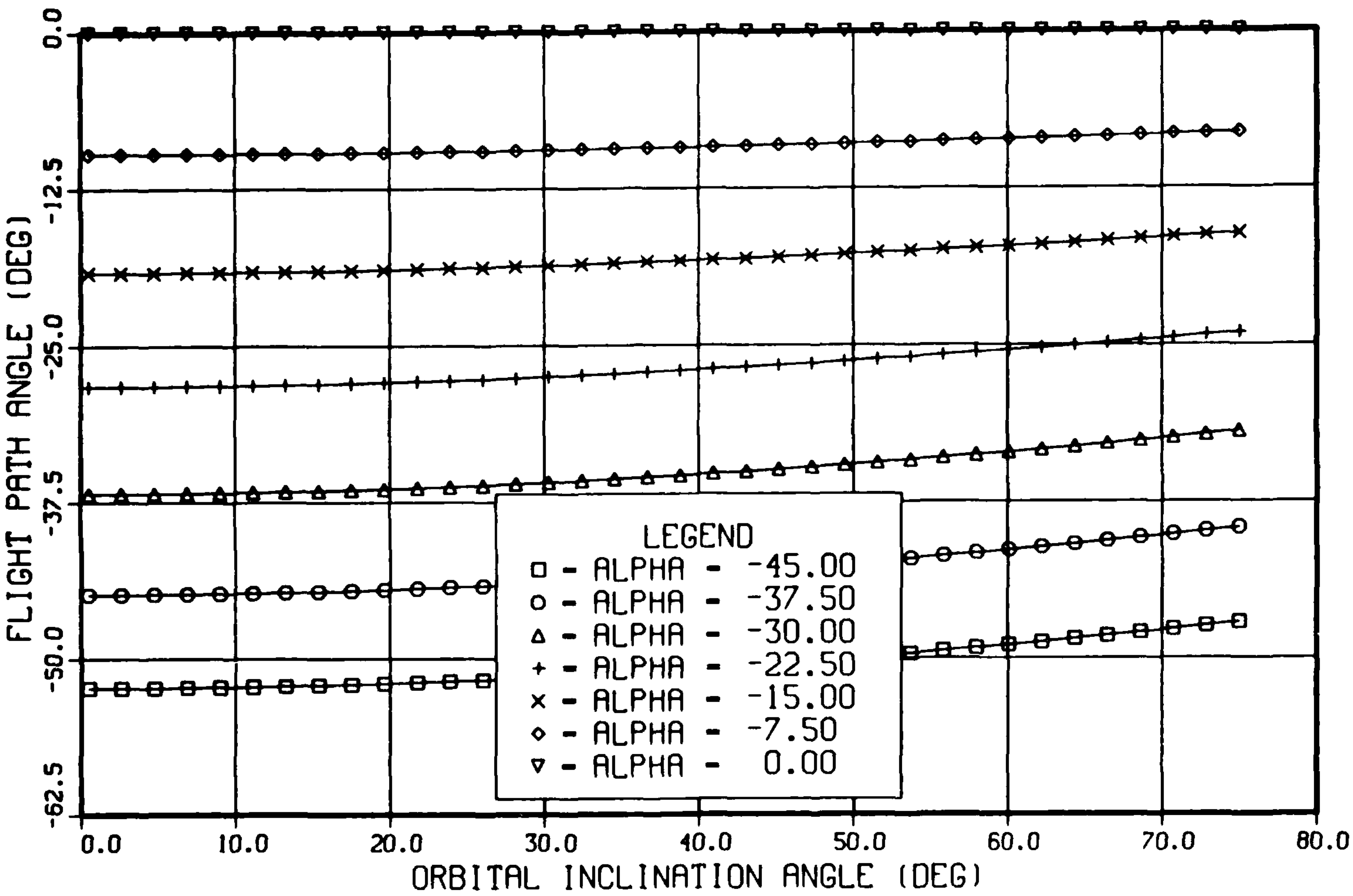

Figure F11. States of Validity for the Non-Rotating Earth, dI/dh Equation of Motion (u = .010 , h = .0047)

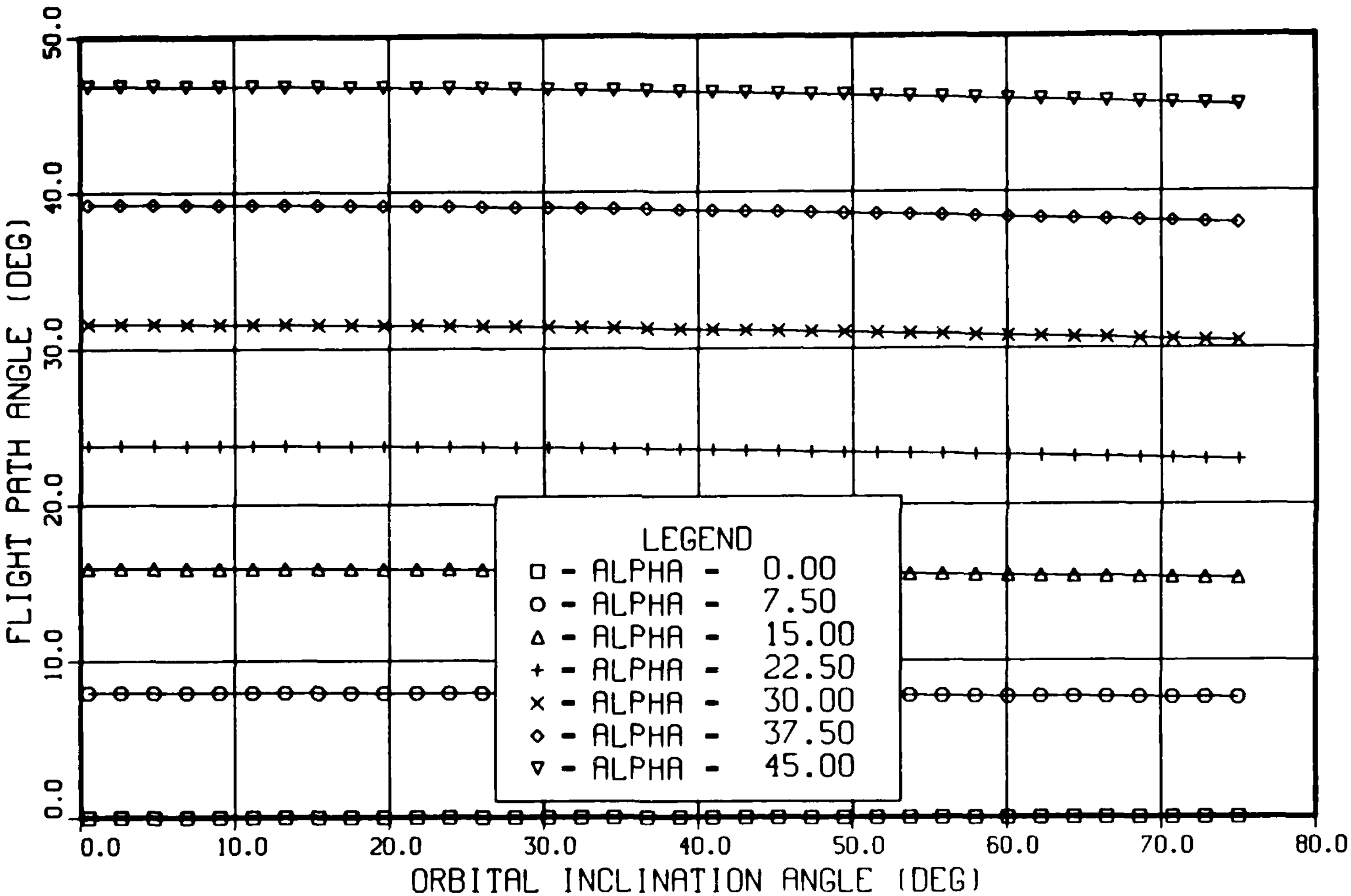

Figure F12. States of Validity for the Non-Rotating Earth, dI/dh Equation of Motion (u = .210 , h = .0077)

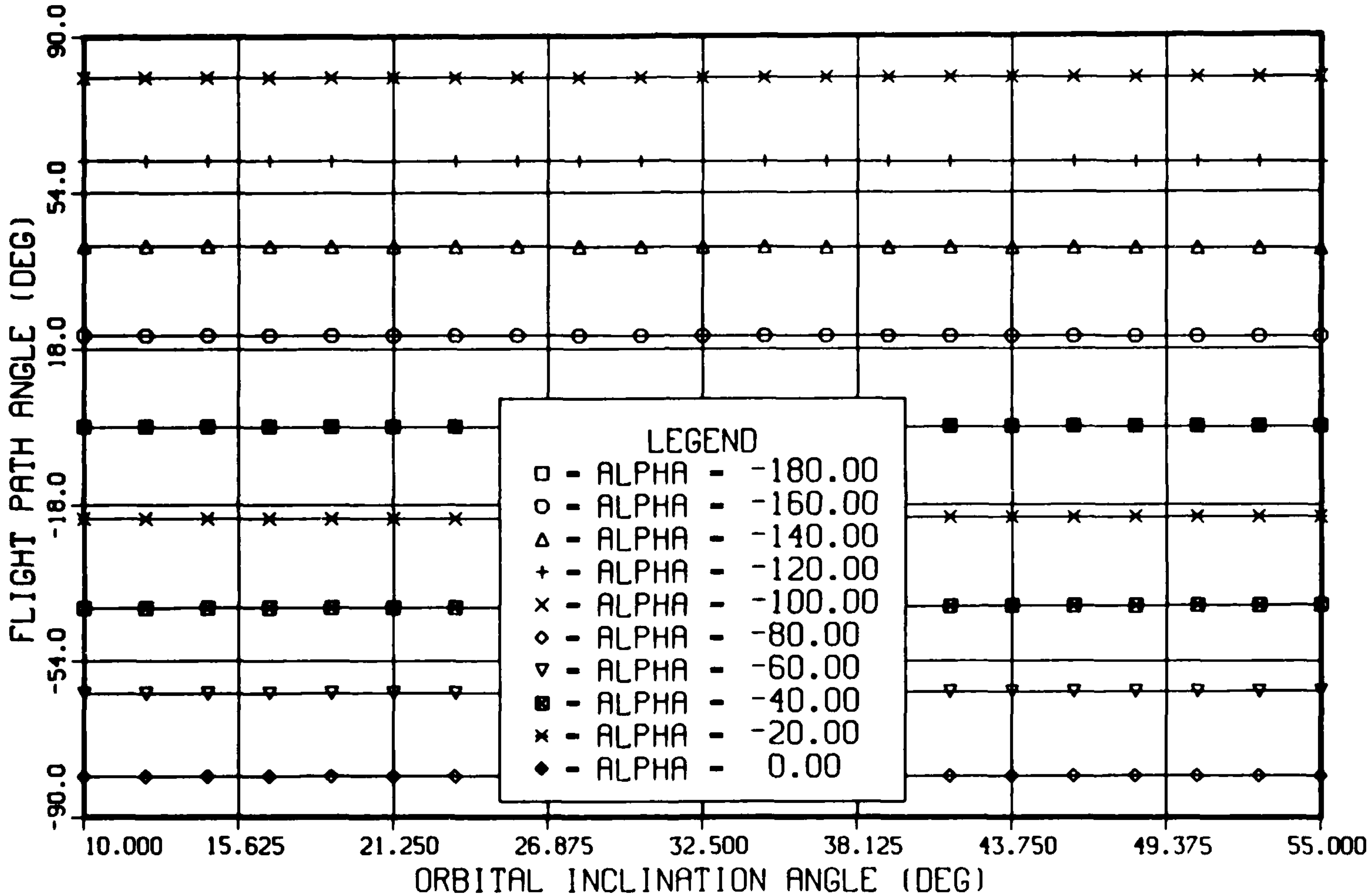

Figure F13. States of Validity for the Non-Rotating Earth, dI/dh Equation of Motion (u = .210 , h = .0077)

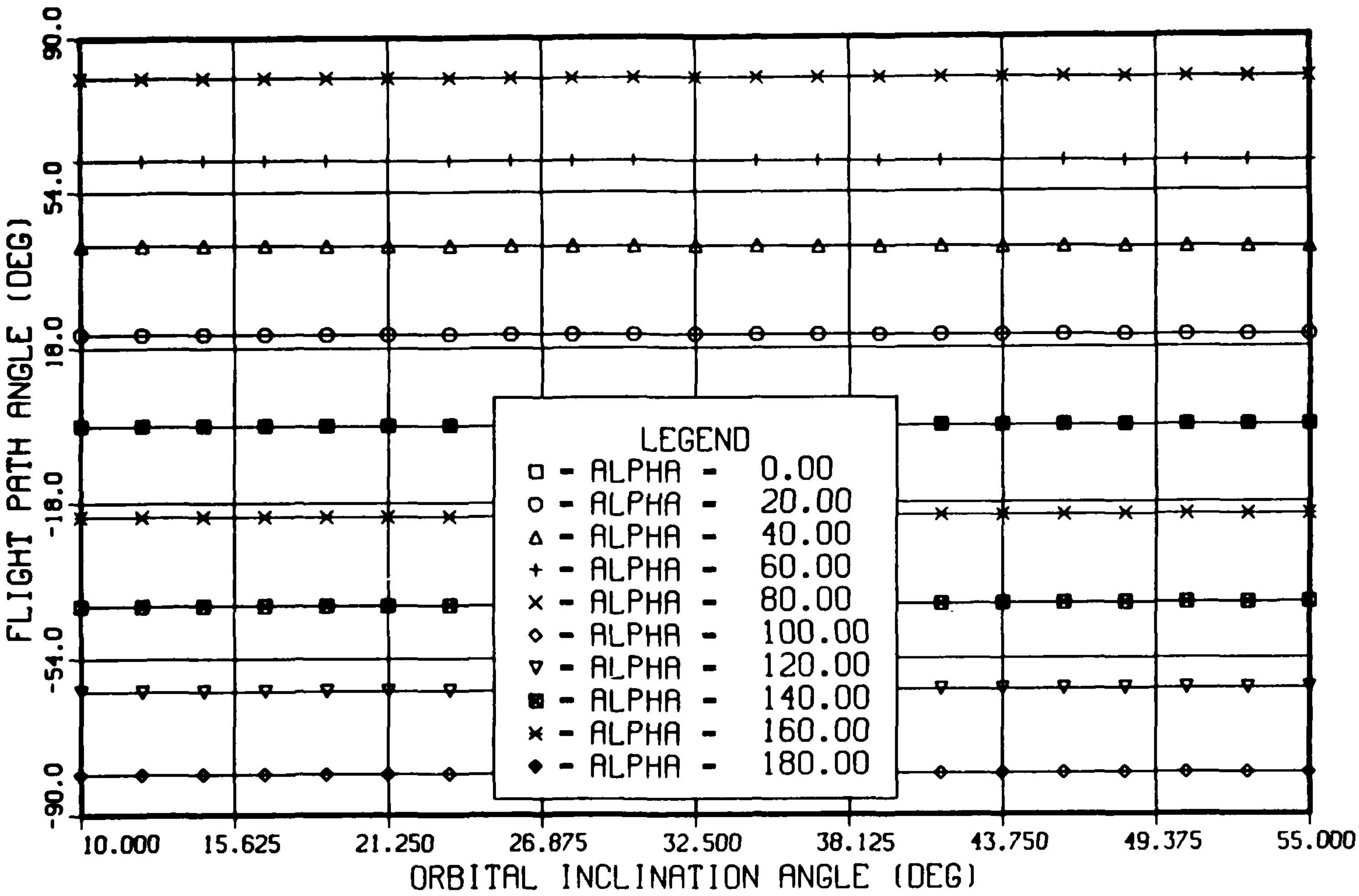

Figure F14. States of Validity for the Non-Rotating Earth, dI/dh Equation of Motion (u = .210 , h = .0077)

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The assumption of a non-rotating planet, common in most analytical entry trajectory analyses, has been shown to produce significant errors in some solutions for the lifting atmospheric entry of Earth. This thesis presents an investigation of the validity of the non-rotating planet assumption for general three-dimensional Earth atmospheric entry.

In this effort, the three-dimensional equations of motion for lifting atmospheric are expanded to include a rotating planet model. A strictly exponential atmosphere, rotating at the same rate as the planet, is assumed with density as a function of radial distance from the planet's surface. Solutions are developed for the non-rotating Earth equations of motion and for one of the rotating Earth equations of motion using the method of matched asymptotic expansions.

It is shown that the non-rotating Earth assumption produces incorrect entry trajectory results for entry orbital inclination angles between 0.5 and 75.0 degrees and vehicle speeds ranging from circular orbital velocities to low supersonic speeds. However, a variety of realistic trajectory states exist where some of the non-rotating Earth equations of motion are valid. Three of the non-rotating equations of motion are found to be valid for the same entry trajectory states. Other, independent trajectory states exist where a fourth non-rotating Earth equation of motion is valid. A fifth equation of motion is never valid for the ranges of orbital inclination angle and speeds investigated. Trends in the results of the trajectory states of validity are discussed and methods to estimate some of these states are presented.